Chirality, helicity and spin

Two component Weyl spinors form two irreducible unitary spin-1/2 representations of the Poincaré group – this means physical particles with spin 1/2 have two degrees of freedom (for example, spin up or spin down).

But our Dirac Lagrangian is most naturally constructed from a four component Dirac spinor, which can be decomposed into the direct sum of irreducible representations of the Lorentz group.

To see how these two pictures are related, we can work in the Weyl representation, for which

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \quad \text{and} \quad S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \quad \text{are block diagonal}$$

This means the upper and lower pairs of components transform separately under a Lorentz transformation

$$\psi \to \psi' = \psi + \frac{1}{2}\begin{pmatrix} (i\theta_i - \beta_i)\sigma_i & 0 \\ 0 & (i\theta_i + \beta_i)\sigma_i \end{pmatrix}\psi$$

and we call these pairs left-handed and right-handed

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

left-handed: transform in $(\frac{1}{2}, 0)$ representation

right-handed: transform in $(0, \frac{1}{2})$ representation

The handedness of the Weyl spinor is called <u>chirality</u>.

The left- and right-handed components can be projected out through the matrix

$$\boxed{\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3}$$

in the Weyl representation

$$\gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

$$\gamma^5\psi = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} -\psi_L \\ \psi_R \end{pmatrix}$$

$\Rightarrow$ $\psi_{L/R}$ are eigenstates of γ^5 with eigenvalues ∓ 1.

Moreover

$$P_R = \frac{1}{2}(1+\gamma^5)$$
$$P_L = \frac{1}{2}(1-\gamma^5)$$

satisfy $P_{R/L}^2 = P_{R/L}$, $P_R P_L = P_L P_R = 0$

allow us to pick out the Weyl spinors

$$P_L\psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix} \qquad P_R\psi = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix}$$

Our new gamma matrix satisfies

$$(\gamma^5)^2 = \mathbb{1}$$

$$\{\gamma^5, \gamma^\mu\} = 0$$

In terms of $\varphi_{L/R}$, the Dirac equation becomes

$$\bar{\sigma}\cdot p\,\varphi_L = m\varphi_R$$
$$\sigma\cdot p\,\varphi_R = m\varphi_L$$

$\leftarrow$ from $\begin{pmatrix} -m & \sigma\cdot p \\ \bar{\sigma}\cdot p & -m \end{pmatrix}\begin{pmatrix} \varphi_L \\ \varphi_R \end{pmatrix} = 0$

Therefore the fermion mass couples left- and right-handed Weyl spinors.

$\uparrow$ or "mixes"

If $m = 0$ then

$$\bar{\sigma}\cdot p\,\varphi_L = 0$$
$$\sigma\cdot p\,\varphi_R = 0$$

and, in fact, these two component spinors are eigenstates of the helicity operator

$\uparrow$ the projection of the angular momentum operator along the direction of the momentum.

ie spin

$$h = \tfrac{1}{2}\epsilon_{ijk}\,p^i S^{jk}$$
$$= \tfrac{1}{2}p_i\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$$

with eigenstates $h = +\frac{1}{2}$ for φ_L and $h = -\frac{1}{2}$ for φ_R.

Helicity is only useful for massless theories (for which you cannot boost past the particle), whereas chirality is Lorentz invariant for massive particles. Chirality is somehow a "fundamental" quantity for particle physics, but helicity is not.

To summarise

<u>Spin</u> - a vector quantity ← unless we mean its eigenvalue!
- eigenvalue of $\sigma^i/2$ for fermions
- fundamental to a particle's identity

<u>Helicity</u> - projection of spin on direction of motion
- eigenvalue of $\frac{\sigma^i p^i}{E}$
- meaningful for massless particles
- exists for any spin
- not "fundamental"

<u>Chirality</u> - exists only for (a, b) representations of the Lorentz group, with $a \neq b$
- Lorentz invariant for massive particles
- fermion mass term mixes chiralities
- "fundamental", but generally only relevant for theories that are not symmetric between ψ_L & ψ_R

in different irreducible representations!

<u>$\psi_{L/R}$ states</u> - do not mix under Lorentz transformations
- each have two components → spin states of the fermion
- eigenstates of helicity in the massless limit

Interpreting spinor solutions

Our spinor solutions have been expressed in terms of the two-component constant spinors ξ and η.

It is often convenient to introduce a basis for these spinors. As an example, we can choose to look at the spin component projected onto the z-axis and, in this basis, we have fermions with spin up and down. We label our choice of basis as ξ^s, η^s.

In our example

$$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Q: what is ξ^2 ?

is an eigenstate of σ^z, so corresponds to spin up along the z-axis.

Our spinor solutions must satisfy

$$\boxed{\begin{aligned} \xi^{r\dagger}\xi^s &= \delta^{rs} \\ \eta^{r\dagger}\eta^s &= \delta^{rs} \end{aligned}}$$

← r and s can be 1 or 2

We are now in a position to interpret our Dirac four spinors:

$$u^s(\bar{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} \qquad v^s(\bar{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta^s \end{pmatrix}$$

In any frame

- $u^s(\bar{p})$ are positive energy spin-$\frac{1}{2}$ fermions with two spin states $(s=1,2)$
- $v^s(\bar{p})$ are positive energy spin-$\frac{1}{2}$ antifermions with two spin states $(s=1,2)$

The final aspect missing from our discussion so far is the normalisation. To determine this, we first need a Lorentz covariant inner product.

If this is surprising, think about $\bar{\varphi}\varphi$ vs $\varphi^\dagger\varphi$.

You might guess that $u^{r\dagger}(\bar{p})\,u^s(\bar{p})$ is suitable, but it is not. We need

$$\bar{u}^r(\bar{p})\,u^s(\bar{p}) = \left(\xi^{r\dagger}\sqrt{p\cdot\sigma},\ \xi^{r\dagger}\sqrt{p\cdot\bar{\sigma}}\right)\underbrace{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}}_{\gamma^0}\begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi^s\\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s\end{pmatrix}$$

$$= 2m\,\delta^{rs}$$

Q: what is $u^{r\dagger}(\bar{p})\,u^s(\bar{p})$

Similarly

$$\bar{v}^r(\bar{p})\,v^s(\bar{p}) = -2m\,\delta^{rs}$$

$$\bar{u}^r(\bar{p})\,v^s(\bar{p}) = \bar{v}^r(\bar{p})\,u^s(\bar{p}) = 0$$

This defines the "conventional" normalisation for massive Dirac spinors.

We also have the <u>outer products</u>

$$\sum_{s=1}^{2} u^s(\bar{p})\,\bar{u}^s(\bar{p}) = \not{p} + m$$

$$\sum_{s=1}^{2} v^s(\bar{p})\,\bar{v}^s(\bar{p}) = \not{p} - m$$

To prove this (or, at least, the first one) we use

$$\sum_{s=1}^{2} u^s(\bar{p})\,\bar{u}^s(\bar{p}) = \sum_{s=1}^{2} \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} \left(\xi^{s\dagger}\sqrt{p\cdot\sigma},\ \xi^{s\dagger}\sqrt{p\cdot\bar{\sigma}} \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{(p\cdot\sigma)(p\cdot\bar{\sigma})} & \sqrt{(p\cdot\sigma)(p\cdot\sigma)} \\ \sqrt{(p\cdot\bar{\sigma})(\bar{p}\cdot\bar{\sigma})} & \sqrt{(p\cdot\bar{\sigma})(p\cdot\sigma)} \end{pmatrix}$$

N.B. $\sum_{s=1}^{2} \xi^s \xi^{s\dagger} = \mathbb{1}$

$$= \begin{pmatrix} m & p_\mu\sigma^\mu \\ p_\mu\bar{\sigma}^\mu & m \end{pmatrix}$$

$$= \not{p} + m$$

Discrete symmetries

The free Dirac Lagrangian is invariant under continuous Lorentz transformations, as we've seen, and under three discrete symmetries:

- charge conjugation ← acts on functions of spacetime
- parity
- time reversal

$\Big\}$ act on spacetime itself, part of the Lorentz group itself

Charge conjugation

$\boxed{\text{Schwartz 11.4}}$
Tong 4.6 P+S 3.6

Charge conjugation acts on Dirac spinors to take particles to antiparticles and flip their spin

$$C: \; \psi(x) \to \psi^c(x) = -i\gamma^2 \psi^*(x)$$

← This is in the Weyl basis

To see how this works in practice, let's take the example of a spin-½ particle in its rest frame, which has two spin states

$$u^\uparrow = \sqrt{m}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} \qquad u^\downarrow = \sqrt{m}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

Then

$$(u^\uparrow)^c = -i\gamma^2\sqrt{m}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$$

$$= -i\sqrt{m}\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&i&0&0\\-i&0&0&0\end{pmatrix}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} = \sqrt{m}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}$$

← this is exactly $v^\downarrow$, the antifermion with spin down, as expected

In general, we can write charge conjugation in the form

$$\psi^{(c)} = C\psi^*$$

$\uparrow$ basis dependent 4×4 matrix with $C^\dagger C = \mathbb{1}$

$C^\dagger \gamma^\mu C = -(\gamma^\mu)^*$

In the Weyl basis we had

$$C = -i\gamma^2$$

but this need not be the case. If we choose

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix} \quad \gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \quad \gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}$$

then

- $(\gamma^\mu)^* = -\gamma^\mu$ $\leftarrow$ γ^μ are imaginary!
- generators of the Lorentz group are real
- $C = \mathbb{1}$
- $\psi^{(c)} = \psi^*$ $\leftarrow$ fermions are their own antifermions

This is the <u>Majorana basis</u>. Majorana spinors are then

$$\psi(x) = \begin{pmatrix} \psi_L(x) \\ -i\sigma^2\psi_L^*(x) \end{pmatrix}$$

Proof: $C(i\not{\partial} - m)\psi = 0$

$\Rightarrow (-i\not{\partial}^* - m)\psi^* = 0$

$\Rightarrow (-iC\gamma^{\mu *}C^\dagger C\partial_\mu - mC\cdot C^\dagger C)\psi^* = 0$

$\Rightarrow (i\not{\partial} - m)C\psi^* = 0$ $\blacksquare$

Whatever basis we work in, if $\psi(x)$ satisfies the Dirac equation, then so does $\psi^{(c)}(x)$

Parity

Schwarz 11.5

The parity operator takes spatial spacetime components to their negative counterparts – a reflection in space.

$$P: x^\mu = (x^0, \bar{x}) \rightarrow x'^\mu = (x^0, -\bar{x})$$ ← note $P^2 = 1$

This is still a Lorentz transformation, but a discrete one, and therefore cannot be described by exponentiation of a generator.

The action of parity on spinors depends on the type of spinors. If we consider the generators in the Weyl basis

$$S(\Lambda)_{rot} = \begin{pmatrix} e^{-\frac{i}{2}\bar{\theta}\cdot\bar{\sigma}} & 0 \\ 0 & e^{-\frac{i}{2}\bar{\theta}\cdot\bar{\sigma}} \end{pmatrix}$$

← to take a Weyl spinor back to itself we have to rotate by 4π!

$$S(\Lambda)_{boost} = \begin{pmatrix} e^{-\frac{1}{2}\bar{\beta}\cdot\bar{\sigma}} & 0 \\ 0 & e^{+\frac{1}{2}\bar{\beta}\cdot\bar{\sigma}} \end{pmatrix}$$

$$\hookrightarrow \varphi_{L/R} \xrightarrow[\text{rotations}]{} e^{-\frac{i}{2}\bar{\theta}\cdot\bar{\sigma}}\varphi_{L/R}$$

$$\varphi_{L/R} \xrightarrow[\text{boosts}]{} e^{\mp\frac{1}{2}\bar{\beta}\cdot\bar{\sigma}}\varphi_{L/R}$$

Since parity flips the sign of boosts, but not rotations, this means parity must transform φ_L to φ_R and vice versa.

To see this another way, we can note $\varphi_{L/R}$ are eigenstates of the helicity operator $\quad \hat{h}\,\varphi_{R/L} = \pm\frac{1}{2}\varphi_{R/L}$

↑ if $m=0$

Now $\hat{h} = \frac{\bar{\sigma} \cdot \bar{p}}{|\bar{p}|}$ and parity commutes with the spin operator, but flips the sign of the three-momentum.

Q: why?

Therefore $P : \psi_{R/L} \to \psi_{L/R}(-\bar{x})$

we cannot assign parity consistently to the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations alone.

For Dirac spinors, parity acts as

$P : \psi \to \gamma^0 \psi$ ← switches $\begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}$ to $\begin{pmatrix}\psi_R \\ \psi_L\end{pmatrix}$

This means that

$$P : \bar{\psi}\psi(t,\bar{x}) \to \bar{\psi}\psi(t,-\bar{x})$$

$$P : \bar{\psi}\gamma^\mu\psi(t,\bar{x}) \to -\bar{\psi}\gamma^\mu\psi(t,-x)$$

$$P : \bar{\psi}\gamma^\mu\gamma^5\psi(t,\bar{x}) \to \begin{cases} -\bar{\psi}\gamma^\mu\gamma^5\psi(t,-\bar{x}) & \mu = 0 \\ \bar{\psi}\gamma^\mu\gamma^5\psi(t,-\bar{x}) & \mu = 1,2,3 \end{cases}$$

These quark bilinears can all be characterised by their behaviour under C, P and T. In fact there are sixteen such structures, and they are collected in a table on page 71 of P+S (along with their properties under CPT).

Their names are

$\bar{\psi}\psi$ - scalar

$\bar{\psi}\gamma^5\psi$ - pseudoscalar

$\bar{\psi}\gamma^\mu\psi$ - vector

$\bar{\psi}\gamma^\mu\gamma^5\psi$ - axial-vector

$\bar{\psi}\sigma^{\mu\nu}\psi$ - tensor

} names come from their transformation properties under Lorentz transformations

Whether a theory is parity invariant depends on the theory itself. Quantum electrodynamics is parity invariant, but the weak nuclear force is not, because it is <u>chiral</u>.

<u>Not</u> symmetric under $\psi_L \leftrightarrow \psi_R$

<u>Time reversal</u>

$\boxed{\text{Schwarz 11.6}}$

Time reversal is exactly what you think it is

$$T: \; x^\mu = (x^0, \bar{x}) \rightarrow x'^\mu = (-x^0, \bar{x})$$

except that it is not. At least in the sense that it does not act on fields as you might expect, because it is <u>antilinear</u> and acts on complex numbers as

$$T: \; a + ib \rightarrow a - ib \quad \text{for } a, b \text{ real.}$$

For more details, see Schwarz

The CPT theorem says that any Lorentz invariant unitary QFT must be invariant under the combined actions of charge conjugation, parity and time reversal.

This means that when experimentalists observe CP violation in weak decays, there is a simultaneous violation of time reversal by exactly the opposite amount.

Searching for new sources of CP violation, beyond the currently known effects in the standard model of particle physics, is an ongoing area of very active research. This is largely because standard model CP violation cannot explain the baryon-antibaryon asymmetry in the universe.

ie matter   antimatter

Sakharov formulated conditions on our universe that are requir for matter-antimatter asymmetry. One of them is a lot of CP violation.