Gamma matrices

Schwarz 10.3

Tong 4.1 and P+S 3.2

We've introduced these 4x4 gamma matrices as a way to simplify notation, but, in fact, they are closely related to the structure of spacetime and the Lorentz group.

The gamma matrices are

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

← this is actually not a definition, but just one representation

and in general they satisfy the Clifford algebra

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}}$$

← this is a definition

note there is an implicit 4x4 identity matrix here

Pauli sigma matrices have different representations — and so do gamma matrices. But the Clifford algebra is a general definition (like the Lie algebra) that any representation must satisfy. One example is the one we've seen — the Weyl representation

$$\gamma^0 = \begin{pmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&-1&0&0 \\ -1&0&0&0 \end{pmatrix} \quad \gamma^2 = \begin{pmatrix} 0&0&0&-i \\ 0&0&i&0 \\ 0&i&0&0 \\ -i&0&0&0 \end{pmatrix} \quad \gamma^3 = \begin{pmatrix} 0&0&1&0 \\ 0&0&0&-1 \\ -1&0&0&0 \\ 0&1&0&0 \end{pmatrix}$$

We now introduce

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

There are many conventions for the $S^{\mu\nu}$! Tong uses $S^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$, while P+S use $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$.

and define

$$S^{\mu\nu} = \frac{1}{2}\sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

which satisfies the Lie algebra of the Lorentz group

$$[S^{\mu\nu}, S^{\rho\sigma}] = i\left(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho}\right)$$

In other words, the $S^{\mu\nu}$ are the generators of the Lorentz group! In this case, in the spinor representation.

These generators are not the same as the $V^{\mu\nu}$ generators (in the vector representation) - the $S^{\mu\nu}$ are complex, although they are still specified by six matrices.

For example, in the Weyl representation

$$S^{ij} = \frac{1}{2}\epsilon_{ijk}\begin{pmatrix}\sigma_k & 0 \\ 0 & \sigma_k\end{pmatrix} \qquad S^{0i} = -\frac{i}{2}\begin{pmatrix}\sigma_i & 0 \\ 0 & -\sigma_i\end{pmatrix}$$

Technically, the $S^{\mu\nu}$ are the generators for $\psi(x) = (\psi_R(x), \psi_L(x))$ and $(S^{\mu\nu})^+$ are the generators for $\psi(x) = (\psi_L(x), \psi_R(x))$. The algebra is the same.

One of my earlier claims was that indices are one of the trickiest parts of QFT. The properties of gamma matrices and spinors are a good example of this.

Note that $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ really means

$$(\gamma^\mu)_{\alpha\beta}(\gamma^\nu)_{\beta\gamma} + (\gamma^\nu)_{\alpha\beta}(\gamma^\mu)_{\beta\gamma} = 2g^{\mu\nu}\mathbb{1}_{\alpha\beta}$$

We want dot products of vectors to be Lorentz invariant, so this means that the spinor and vector representations have related Lorentz transformations, because

$$\begin{aligned} v \cdot v &= v_\mu v^\mu \\ &= v_\mu g^{\mu\nu} v_\nu \\ &= v_\mu \tfrac{1}{2}\{\gamma^\mu, \gamma^\nu\} v_\nu \end{aligned}$$

In fact

$$\Lambda_s^{-1}\gamma^\mu\Lambda_s = (\Lambda_v)^{\mu\nu}\gamma_\nu$$

Lorentz transformation acting on vector indices

Lorentz transformation acting on spinor indices

follows from $\bar{\psi}\gamma^\mu\psi \to (\Lambda_v)^{\mu\nu}\bar{\psi}\gamma_\nu\psi$

We can also deduce

$$(\gamma^0)^2 = \mathbb{1} \quad \text{and} \quad (\gamma^i)^2 = -\mathbb{1}$$

follows from $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

and that

$$(\gamma^0)^\dagger = \gamma^0 \quad \text{and} \quad (\gamma^i)^\dagger = -\gamma^i$$

This means that

$$(S^{\mu\nu})^\dagger = \left(\frac{i}{4}[\gamma^\mu, \gamma^\nu]\right)^\dagger$$

$$= -\frac{i}{4}[\gamma^{\nu\dagger}, \gamma^{\mu\dagger}]$$

$$= \frac{i}{4}[\gamma^{\mu\dagger}, \gamma^{\nu\dagger}] \quad \Rightarrow \quad S^{ij\dagger} = S^{ij} \quad \text{and} \quad S^{0i\dagger} = -S^{0i}$$

rotations are unitary

boosts are not unitary

Moreover

$$(\gamma^i)^\dagger = -\gamma^i = \gamma^0\gamma^i\gamma^0 \quad \text{and} \quad (\gamma^0)^\dagger = \gamma^0 = \gamma^0\gamma^0\gamma^0$$

so $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$

From this we can also prove that

$$\gamma^0(S^{\mu\nu})^\dagger\gamma^0 = \gamma^0\frac{i}{4}[\gamma^{\mu\dagger}, \gamma^{\nu\dagger}]\gamma^0$$

$$= \frac{i}{4}[\gamma^0\gamma^{\mu\dagger}\gamma^0, \gamma^0\gamma^{\nu\dagger}\gamma^0]$$

$$= \frac{i}{4}[\gamma^\mu, \gamma^\nu] = S^{\mu\nu}$$

$$\Rightarrow (\gamma^0\Lambda_s\gamma^0)^\dagger = \gamma^0 \exp[i\theta_{\mu\nu}S^{\mu\nu}]^\dagger\gamma^0$$

$$= \exp[-i\theta_{\mu\nu}\gamma^0 S^{\mu\nu\dagger}\gamma^0]$$

$$= \exp[-i\theta_{\mu\nu}S^{\mu\nu}] = \Lambda_s^{-1}$$

Finally, this allows us to show that $\psi^\dagger \gamma^0 \psi$ is Lorentz invariant

$$\psi^\dagger \gamma^0 \psi \to (\psi^\dagger \Lambda_S^\dagger)\gamma^0(\Lambda_S \psi) = \psi^\dagger \gamma^0 \Lambda_S^{-1}\Lambda_S \psi$$
$$= \psi^\dagger \gamma^0 \psi$$

Then we recall that $\bar{\psi}\psi$ was also Lorentz invariant and we see that $\bar{\psi} = \psi^\dagger \gamma^0$!!

which you can check explicitly.

Let's summarise our gamma matrix stuff

- $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$
- $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$ $\quad \sigma^\mu = (1, \sigma^i) \quad \bar{\sigma}^\mu = (1, -\sigma^i)$
- $(\gamma^0)^2 = \mathbb{1} \quad (\gamma^i)^2 = -\mathbb{1}$
- $(\gamma^0)^\dagger = \gamma^0 \quad (\gamma^i)^\dagger = -\gamma^i$
- $(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0$
- $\Lambda_S^{-1}\gamma^\mu \Lambda_S = (\Lambda_V)^{\mu\nu}\gamma_\nu \quad \Lambda_S^{-1} = (\gamma^0 \Lambda_S \gamma^0)^\dagger$
- $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] = \frac{1}{2}\sigma^{\mu\nu} \quad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$
- $(S^{ij})^\dagger = S^{ij} \quad (S^{0i})^\dagger = -S^{0i}$
- $S^{\mu\nu} = \gamma^0 (S^{\mu\nu})^\dagger \gamma^0$

(a) "Weyl basis"

$$\gamma^0 = \begin{pmatrix} 0_{2\times2} & 1_{2\times2} \\ 1_{2\times2} & 0_{2\times2} \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0_{2\times2} & \sigma^i \\ -\sigma^i & 0_{2\times2} \end{pmatrix}$$

There are no unitary representations of the Lorentz group

$$\overset{1/2}{J}{}^{0i} = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$$

- generates boosts
- block diagonal
- not Hermitian
- $S(\Lambda_{boost})$ not unitary

$$\overset{1/2}{J}{}^{ij} = \frac{\epsilon_{ijk}}{2}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$$

- generates rotations
- block diagonal
- Hermitian
- $S(\Lambda_{rot})$ unitary

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & 1_{2\times2} \end{pmatrix}$$

- block diagonal

(b) "Dirac basis"

γ^5 intricately associated with chiral symmetry

$$\gamma^0 = \begin{pmatrix} 1_{2\times2} & 0_{2\times2} \\ 0 & -1_{2\times2} \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0_{2\times2} & \sigma^i \\ -\sigma^i & 0_{2\times2} \end{pmatrix}$$

$$\overset{1/2}{J}{}^{0i} = \frac{i}{2}\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

- generates boosts
- not block diagonal
- not Hermitian

$$\overset{1/2}{J}{}^{ij} = \frac{\epsilon_{ijk}}{2}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$$

- generates rotations
- block diagonal
- Hermitian

$$\gamma^5 = \begin{pmatrix} 0_{2\times2} & 1_{2\times2} \\ 1_{2\times2} & 0_{2\times2} \end{pmatrix}$$

- not block diagonal

Free particle solutions of the Dirac equation

Schwarz 11.2
Tong 4.3
P+S 3.2 and 3.3

To solve the Dirac equation, we first note that any solution of the Dirac equation is also a solution of the Klein-Gordon equation:

$$(i\not{\partial} - m)\psi = 0 \Rightarrow (i\not{\partial} + m)(i\not{\partial} - m)\psi = 0$$

$$(-\not{\partial}\not{\partial} + im\not{\partial} - im\not{\partial} - m^2)\psi = 0$$

$$-(\partial^2 + m^2)\psi = 0$$

the Klein-Gordon equation!

Here we used

$$\not{a}\not{a} = a_\mu a_\nu \gamma^\mu \gamma^\nu$$

$$= a_\mu a_\nu (2g^{\mu\nu} - \gamma^\nu\gamma^\mu)$$

$$= 2a^2 - \not{a}\not{a} \Rightarrow \not{a}\not{a} = a^2$$

for any four-vector a^μ

This means we can write the solution in the general form

$$\psi(x) = \int \frac{d^3\bar{p}}{(2\pi)^3} u(\bar{p}) e^{-ip\cdot x}$$

with $p^0 = \sqrt{\bar{p}^2 + m^2} > 0$

Here $u(p)$ is a four-component spinor.

Working in the Weyl representation the Dirac equation is

$$\begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar{\sigma} & -m \end{pmatrix} u(\bar{p}) = 0$$

which simplifies in the rest frame, $\bar{p} = 0$, to

$$m\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} u(\bar{p}) = 0 \Rightarrow \text{solutions are constants!}$$

Recall that Weyl spinors have two components, so we usually write these constant solutions as

$$\varphi(\bar{p}=\bar{0}) = \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

one example is to choose $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ which we will interpret as spin up or spin down positive energy solutions

To solve the Dirac equation more generally, we solve in a boosted frame, for example $p^\mu = (E, 0, 0, p^z)$

$$\Rightarrow \quad \sigma\cdot p = \begin{pmatrix} E-p^z & 0 \\ 0 & E+p^z \end{pmatrix} \qquad \bar{\sigma}\cdot\bar{p} = \begin{pmatrix} E+p^z & 0 \\ 0 & E-p^z \end{pmatrix}$$

and use

$$m^2 = E^2-(p^z)^2 = (E-p^z)(E+p^z)$$

to write

$$\begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar{\sigma} & -m \end{pmatrix} = \begin{pmatrix} -\sqrt{(E-p^z)(E+p^z)} & 0 & E-p^z & 0 \\ 0 & -\sqrt{(E-p^z)(E+p^z)} & 0 & E+p^z \\ E+p^z & 0 & -\sqrt{(E-p^z)(E+p^z)} & 0 \\ 0 & E-p^z & 0 & -\sqrt{(E-p^z)(E+p^z)} \end{pmatrix}$$

and then one can check that

$$u(\bar{p}) = \begin{pmatrix} \begin{pmatrix} \sqrt{E-p^z} & 0 \\ 0 & \sqrt{E+p^z} \end{pmatrix}\xi \\ \begin{pmatrix} \sqrt{E+p^z} & 0 \\ 0 & \sqrt{E-p^z} \end{pmatrix}\xi \end{pmatrix}$$

is indeed a solution of

$$\begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar{\sigma} & -m \end{pmatrix} u(\bar{p}) = 0$$

More generally we can write

$$u(\bar{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix}$$

There is a second set of solutions

$$\chi(x) = \int \frac{d^3\bar{p}}{(2\pi)^3}\, v(\bar{p})\, e^{ip\cdot x}$$

with $p^0 = \sqrt{\bar{p}^2 + m^2} > 0$, that satisfy

$$\begin{pmatrix} -m & -p\cdot\sigma \\ -p\cdot\bar{\sigma} & -m \end{pmatrix} v(\bar{p}) = 0$$

which has solutions

$$v(\bar{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta \end{pmatrix}$$

η is a <u>different</u> two component constant spinor.

Q: what is the square root of a matrix?

OK - this is great! We have some solutions of the free Dirac equation, but what do we do with them?

Answering this requires a short detour into understanding chirality, helicity and spin.