Learning outcomes - embedding particles in fields

You will be able to:

- write down the classical Lagrangian for a scalar field
- identify spinor representations of the Lorentz group
- define Weyl spinors
- write down the Dirac Lagrangian and derive the corresponding equation of motion
- define the Clifford algebra
- write down and manipulate gamma matrices
- derive properties of gamma matrices independent of representation
- solve the free Dirac equation and interpret various spinor solutions in terms of particles and their spins
- define chirality, helicity and spin and relate them to the properties of Weyl and Dirac fermions
- calculate and apply inner and outer products for spinors
- define CPT operations
- categorise spinor bilinears according to their CPT properties

Classifying particles and their Lagrangians

Schwartz 8.2

With all this discussion of group theory in mind, let's now start classifying particles and writing down their Lagrangians. Keep in mind we still haven't quantised yet!

Spin 0

particle property (Poincaré group)

For $j = 0$, there is only one degree of freedom for any mass, so we put this into the $(0,0)$ representation – the scalar field

field property (Lorentz group)

Lagrangian must be a Lorentz scalar, but constructing scalars out of scalars is not so hard. The Lagrangian that gives the Klein-Gordon equation is

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$$\mathcal{L}(x) = \frac{1}{2}\partial_\mu \phi(x)\partial^\mu \phi(x) - \frac{1}{2}m^2\phi(x)^2$$

↳ equation of motion is $(\partial^2 + m^2)\phi(x) = 0$

with solutions $\phi(x) = e^{\pm ipx}$ $\quad p^2 = m^2$

Spin ½

Technically, SO(1,3) has no spinor representations. The complexified Lie algebra $so(1,3)_{\mathbb{C}}$ is isomorphic to $su(2) \oplus su(2)$. Spinors are a representation of $SL(2,\mathbb{C})$

There are two representations of the Lorentz group that give spin $j = \frac{1}{2}$ – $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. Unlike the spinor case, we can "just" write down the Lagrangian for these fields though. We have a fair bit of work to do to get to that part where we can "just" write down the Lagrangian.

First let's look at the spinor representations themselves.

Spinor representations

Schwartz 10.2

We want to find 2×2 matrices that satisfy ← because they act on objects with $2j+1 = 2\cdot\frac{1}{2}+1 = 2$ degrees of freedom

$$[J_i^{\pm}, J_j^{\pm}] = i\epsilon_{ijk} J_k^{\pm}$$

$$[J_i^{+}, J_j^{-}] = 0$$

We actually already know what these could be! The Pauli sigma matrices will do the job ← $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$

If we choose

$$\left.\begin{aligned} J_i^- &= \tfrac{1}{2}\sigma_i \\ J_i^+ &= 0 \end{aligned}\right\} (\tfrac{1}{2}, 0) \text{ representation} \quad \leftarrow \text{Re}$$

$$\left.\begin{aligned} J_i^- &= 0 \\ J_i^+ &= \tfrac{1}{2}\sigma_i \end{aligned}\right\} (0, \tfrac{1}{2}) \text{ representation}$$

Then we have found two sets of generators that transform in the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations of the Lorentz group and the objects they act on will represent particles with spin $\frac{1}{2}$.

Given these generators, the "actual" Lorentz transformations will be

$$J_i = J_i^- + J_i^+ \qquad \left\{ \begin{array}{ll} (\frac{1}{2},0) & J_i = \frac{1}{2}\sigma_i + 0 = \frac{1}{2}\sigma_i \\ (0,\frac{1}{2}) & J_i = 0 + \frac{1}{2}\sigma_i = \frac{1}{2}\sigma_i \end{array} \right\} \begin{array}{c} \text{Hermitian} \\ \sigma_i^\dagger = \sigma_i \end{array}$$

$$K_i = i(J_i^- - J_i^+) \qquad \left\{ \begin{array}{ll} (\frac{1}{2},0) & K_i = i(\frac{1}{2}\sigma_i - 0) = \frac{i}{2}\sigma_i \\ (0,\frac{1}{2}) & K_i = i(0 - \frac{1}{2}\sigma_i) = -\frac{i}{2}\sigma_i \end{array} \right\}$$

anti-Hermitian

What are the objects that these transformations act on?

They are spinors!

Spinors are elements of a vector space, but they are not vectors in the Lorentz transformation sense.

More specifically

- $(\frac{1}{2},0)$ representation acts on left-handed Weyl spinors
- $(0,\frac{1}{2})$ representation acts on right-handed Weyl spinors

Weyl spinor indicates they are two-component objects.

Under rotations and boosts, these Weyl spinors behave as

$$\varphi_L \to \varphi_L' = e^{\frac{1}{2}(i\theta_j \sigma_j - \beta_j \sigma_j)} \varphi_L$$

$$\varphi_R \to \varphi_R' = e^{\frac{1}{2}(i\theta_j \sigma_j + \beta_j \sigma_j)} \varphi_R$$

Expanding the exponentials for infinitesimal transformations

$$\delta\varphi_L = \frac{1}{2}(i\theta_j - \beta_j)\sigma_j \varphi_L$$

$$\delta\varphi_R = \frac{1}{2}(i\theta_j + \beta_j)\sigma_j \varphi_R$$

Spinor Lagrangians

Schwarz 10.2.2

Tong 4.2 and P+S 3.5

We have found candidates for fields that encode spin $\frac{1}{2}$ particles – two component Weyl spinors. Now we need to construct a Lagrangian for our theory. One way to do this is to work backwards from the Dirac equation.

The equation of motion for a spin $\frac{1}{2}$ particle.

But we will follow Schwarz and work the other way, based on arguments from Lorentz symmetry.

Lagrangian should

- be Lorentz invariant
- have two degrees of freedom
- be real-valued

By considering the infinitesimal Lorentz transformations, we can show that

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- $\varphi_R^+ \varphi_R$ and $\varphi_L^+ \varphi_L$ are not Lorentz invariant
- $\varphi_L^+ \varphi_R$ and $\varphi_R^+ \varphi_L$ are Lorentz invariant

Q: convince yourself this is true – see Schwartz 10.2.2

- $\left(\varphi_R^+ \varphi_R + \varphi_R^+ \bar{\sigma} \varphi_R\right)$ and $\left(\varphi_L^+ \varphi_L - \varphi_L^+ \bar{\sigma} \varphi_L\right)$

  transform as vectors

introduce $\sigma^\mu = (1, \bar{\sigma})$ $\quad \bar{\sigma}^\mu = (1, -\bar{\sigma})$

so that these can be written as

$\varphi_R^+ \sigma^\mu \varphi_R$ and $\varphi_L^+ \bar{\sigma}^\mu \varphi_L$

$\Rightarrow \varphi_R^+ \sigma^\mu \partial_\mu \varphi_R$ and $\varphi_L^+ \bar{\sigma}^\mu \partial_\mu \varphi_L$ are Lorentz invariant

N.B. The more "obvious" combinations $\partial_\mu(\varphi_R^+ \sigma^\mu \varphi_R)$ and $\partial_\mu(\varphi_L^+ \bar{\sigma}^\mu \varphi_L)$ are total derivatives – so do not make good candidates for a Lagrangian term

Putting these together

$$\mathcal{L} = \underset{\substack{\text{makes term}\\\text{Hermitian}}}{i}\underbrace{\left(\varphi_R^+ \sigma^\mu \partial_\mu \varphi_R + \varphi_L \bar{\sigma}^\mu \partial_\mu \varphi_L\right)}_{\text{kinetic term}} - m\underbrace{\left(\overbrace{\varphi_R^+ \varphi_L + \varphi_L^+ \varphi_R}^{\text{Hermitian conjugates to ensure it is real}}\right)}_{\text{"Dirac mass" term}}$$

We can make this more compact by defining the four-component <u>Dirac spinor</u>

$$\psi(x) = \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix}$$

and its conjugate ← but <u>not</u> $\psi^\dagger(x)$

$$\bar{\psi}(x) = \left(\psi_R^\dagger(x) \quad \psi_L^\dagger(x) \right)$$

and then introduce the 4x4 <u>gamma matrices</u>

$$\gamma^\mu = \begin{pmatrix} 0 & \overbrace{\sigma^\mu}^{2\times 2} \\ \bar{\sigma}^\mu & \underbrace{0}_{2\times 2} \end{pmatrix}$$

← This is a four-vector of 4x4 matrices $\gamma^\mu = (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$

Finally, if we introduce the slash notation

$$\not{a} = a^\mu \gamma_\mu = a_\mu \gamma^\mu$$

then we obtain the <u>Dirac Lagrangian</u>

$$\boxed{\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi}$$

← when we quantise this, it will represent free spin-$\frac{1}{2}$ particles.

The (classical) equation of motion that follows from this Lagrangian is the <u>Dirac equation</u>

Aside

This is not the typical derivation of the Dirac equation that is often presented in the context of relativistic quantum mechanics, which can be summarised as

- Schrödinger equation can be "derived" by replacing the momentum by its operator in the nonrelativistic dispersion relation $E = \frac{p^2}{2m} \rightarrow \hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$

  $\hat{E} = i\frac{\partial}{\partial t}$ and $\hat{p} = -i\hbar\frac{\partial}{\partial x}$

- Attempting the same trick with the relativistic dispersion relation $E^2 = \bar{p}^2 + m^2$ leads to several issues, most notably negative energy solutions, which are hard to interpret.

- Trick is to restrict ourselves to first order derivatives and search for coefficients that satisfy

  $(E - \bar{\alpha}\cdot\bar{p} - \beta m)\psi = 0$ with $E^2 = \bar{p}^2 + m^2$

  $\Rightarrow \{\alpha_i, \alpha_j\} = 2\delta_{ij} \qquad \{\alpha_i, \beta\} = 0$ ← note anticommutator

  ↑ this turns out to be the algebra satisfied by the gamma matrices – the <u>Clifford algebra</u>