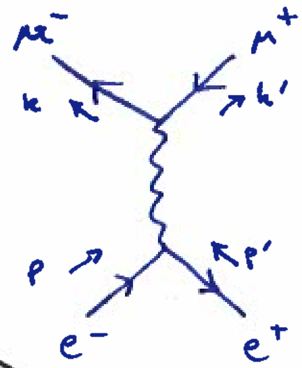
Reminder

$e^+e^- \to \mu^+\mu^-$



1. Feynman rules
2. Invariant matrix element, $i\mathcal{M}$
3. $|\mathcal{M}|^2$
4. Choose reference frame
5. calculate observable

$$i\mathcal{M} = \frac{ig^2}{(p+p')^2} \bar{v}^{s'}(\bar{p}')\gamma^\mu u^s(\bar{p}) \cdot \bar{u}^r(\bar{k})\gamma_\mu v^{r'}(\bar{k}')$$

$$\overline{|\mathcal{M}|^2} = \frac{8g^4}{(p+p')^4}\left[p\cdot k\, p'\cdot k' + p\cdot k'\, p'\cdot k + m_e^2\, k\cdot k' + m_\mu^2\, p\cdot p' + 2m_e^2 m_\mu^2\right]$$

So now we need to move onto $\boxed{4.}$

So now all we need to do is calculate a bunch of traces...

Thankfully there are a few tricks we can use.

→ the "<u>trace identities</u>"

These identities follow from the properties of γ matrices

↑ one reason I summarised them in one place earlier

- $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$
- $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$
- $\{\gamma^\mu, \gamma^5\} = 0$
- $(\gamma^5)^2 = \mathbb{1}$

Identities

1. $\mathrm{Tr}(\gamma^\mu) = 0$
2. $\mathrm{Tr}(\gamma^{\mu_1} \dots \gamma^{\mu_n}) = 0$ if n odd
3. $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$
4. $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$
5. $\mathrm{Tr}(\gamma^5) = 0$
6. $\mathrm{Tr}(\gamma^5\gamma^{\mu_1} \dots \gamma^{\mu_n}) = 0$ for $n \leq 3$
7. $\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4i\epsilon^{\mu\nu\rho\sigma}$

Let's prove a couple of others and leave the rest for HW.

1. $$\mathrm{Tr}(\gamma^\mu) = \mathrm{Tr}(\gamma^5\gamma^5\gamma^\mu) \overset{\text{cyclic}}{=} \mathrm{Tr}(\gamma^5\gamma^\mu\gamma^5)$$
$$= -\mathrm{Tr}(\gamma^\mu\gamma^5\gamma^5) = -\mathrm{Tr}(\gamma^\mu) \quad \Rightarrow \quad = 0$$

↑ similar argument works for n odd

3. $$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = \mathrm{Tr}(2g^{\mu\nu} - \gamma^\nu\gamma^\mu)$$
$$= 2g^{\mu\nu}\,\mathrm{Tr}\,\mathbb{1} - \mathrm{Tr}(\gamma^\nu\gamma^\mu)$$
$$= 8g^{\mu\nu} - \mathrm{Tr}(\gamma^\mu\gamma^\nu)$$
$$\Rightarrow 2\,\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 8g^{\mu\nu}$$
$$\Rightarrow \mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$$

repeated application of similar ideas works for $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)$

Now let's use these tricks in our invariant matrix element

$$\mathrm{Tr}\left[(\not{p}+m_e)\gamma^\mu(\not{p}'-m_e)\gamma^\nu\right] = \mathrm{Tr}\left[\not{p}\,\gamma^\mu\not{p}'\gamma^\nu + m_e\gamma^\mu\not{p}'\gamma^\nu + \not{p}\,\gamma^\mu(-m_e)\gamma^\nu + m_e\gamma^\mu(-m_e)\gamma^\nu\right]$$
$$= p_\alpha p'_\beta\,\mathrm{Tr}\left[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\right] + m_e p'_\alpha\,\mathrm{Tr}\left[\gamma^\mu\gamma^\alpha\gamma^\nu\right] - m_e p_\alpha\,\mathrm{Tr}\left[\gamma^\alpha\gamma^\mu\gamma^\nu\right] - m_e^2\,\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\right]$$
$$= p_\alpha p'_\beta\, 4\left(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\nu}g^{\mu\beta}\right) - m_e^2\, 4g^{\mu\nu}$$
$$= 4\left(p^\mu p'^\nu - p\cdot p' g^{\mu\nu} + p^\nu p'^\mu - m_e^2 g^{\mu\nu}\right)$$
$$= 4\left(p^\mu p'^\nu + p^\nu p'^\mu - (p\cdot p' + m_e^2)\,g^{\mu\nu}\right)$$

Similarly we get

$$\mathrm{Tr}\left[(\not{k}'-m_{(\mu)})\gamma_\mu(\not{k}+m_{(\mu)})\gamma_\nu\right] = 4\left(k_\mu k'_\nu + k_\nu k'_\mu - (k\cdot k' + m_{(\mu)}^2)\,g_{\mu\nu}\right)$$

Thus our invariant matrix element is

$$\overline{|\mathcal{M}|^2} = \frac{1}{4}\frac{g^4}{(p+p')^4}\, 4\left(p^\mu p'^\nu + p^\nu p'^\mu - g^{\mu\nu}(p\cdot p' + m_e^2)\right) \times 4\left(k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}(k\cdot k' + m_{(\mu)}^2)\right)$$

$$\overline{|M|^2} = \frac{4g^4}{(p+p')^4}\Big[\, p\cdot k\, p'\cdot k' + p\cdot k'\, p'\cdot k - p\cdot p'\left(k\cdot k' + m_{(\mu)}^2\right)$$
$$+ p'\cdot k\, p\cdot k' + p\cdot k\, p'\cdot k' - p\cdot p'\left(k\cdot k' + m_{(\mu)}^2\right)$$
$$- 2k\cdot k'\left(p\cdot p' + m_e^2\right) + 4\left(p\cdot p' + m_e^2\right)\left(k\cdot k' + m_{(\mu)}^2\right)\Big]$$

$\uparrow\ g^{\mu\nu}g_{\mu\nu} = g^{\mu}{}_{\mu} = 4$

$$= \frac{4g^4}{(p+p')^4}\Big[2p\cdot k\, p'\cdot k' + 2p\cdot k'\, p'\cdot k - 2p\cdot p'\left(k\cdot k' + m_{(\mu)}^2\right)$$
$$- 2k\cdot k'\left(p\cdot p' + m_e^2\right) + 4\left(p\cdot p' + m_e^2\right)\left(k\cdot k' + m_{(\mu)}^2\right)\Big]$$

$$= \frac{4g^4}{(p+p')^4}\Big[2p\cdot k\, p'\cdot k' + 2p\cdot k'\, p'\cdot k + 2m_{(\mu)}^2\, p\cdot p' + 2m_e^2\, k\cdot k' + 4m_e^2 m_{(\mu)}^2\Big]$$

$$= \frac{8g^4}{(p+p')^4}\Big[(p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) + m_e^2\, k\cdot k' + m_{(\mu)}^2\, p\cdot p' + 2m_e^2 m_{(\mu)}^2\Big]$$

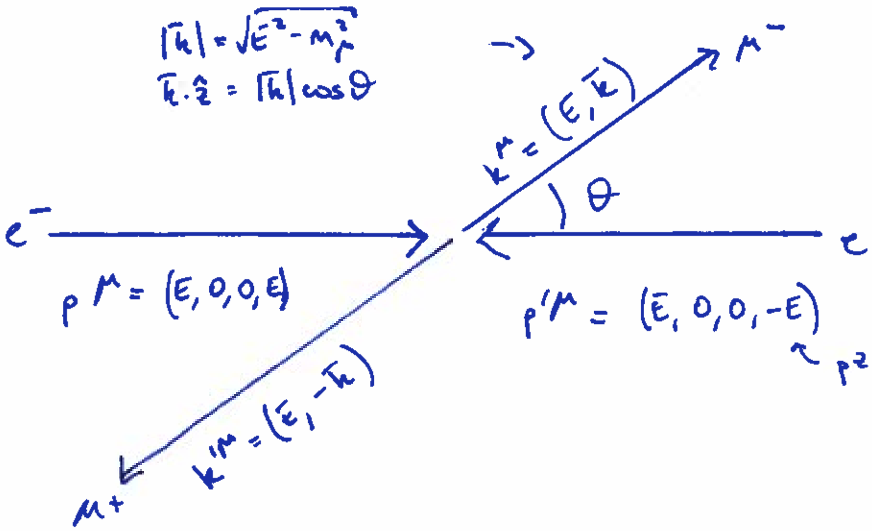
↑ N.B. it's Lorentz invariant!

4. Now we choose a convenient reference frame, in particular we will choose the <u>centre of momentum frame</u> (CM) $\Rightarrow$ $\boxed{\sum \bar{p} = 0}$

We will also simplify our lives by assuming $\frac{m_e}{E} \ll 1$.

↑ recall $m_e \sim 0.5$ MeV!

Then our collision looks like



$\nwarrow p^z = E$ because $m_e \simeq 0$

Q: What energy do we need for this error to be $O(g^4)$? ~ 70 MeV

Now we can simplify the scalar products

$$(p+p')^2 = (2E)^2 = 4E^2$$
$$p \cdot p' = E^2 - E(-E) = 2E^2$$
$$p \cdot k = E^2 - E|\bar{k}|\cos\vartheta$$
$$p' \cdot k' = E^2 - E|\bar{k}|\cos\vartheta$$
$$p \cdot k' = E^2 + E|\bar{k}|\cos\vartheta$$
$$p' \cdot k = E^2 + E|\bar{k}|\cos\vartheta$$

We plug these into our invariant matrix element

$$\overline{|M|^2} = \frac{8g^4}{(4E^2)^2}\left[(E^2 - E|\bar{k}|\cos\vartheta)^2 + (E^2 + E|\bar{k}|^2\cos\vartheta)^2 + m_{(\mu)}^2 \cdot 2E^2\right]$$

$$= \frac{g^4}{2E^4} E^2 \left[(E - |\bar{k}|\cos\vartheta)^2 + (E + |\bar{k}|\cos\vartheta)^2 + 2m_{(\mu)}^2\right]$$

$$= \frac{g^4}{2E^2}\left[2E^2 + 2|\bar{k}|^2\cos^2\vartheta + 2m_{(\mu)}^2\right]$$

$$= g^4\left[1 + \frac{m_{(\mu)}^2}{E^2} + \frac{|\bar{k}|^2\cos^2\vartheta}{E^2}\right] \qquad \text{\& } |\bar{k}|^2 = E^2 - m_{(\mu)}^2$$

$$= g^4\left[1 + \frac{m_{(\mu)}^2}{E^2} + \left(1 - \frac{m_{(\mu)}^2}{E^2}\right)\cos^2\vartheta\right]$$

$\boxed{5.}$ Finally we can substitute our expression for $\overline{|M|^2}$ into

$$\left.\frac{d\sigma}{d\Omega}\right|_{cm} = \frac{1}{4E_p E_{p'}} \frac{1}{|v_p - v_{p'}|} \frac{1}{16\pi^2} \frac{|\bar{k}|}{E_{cm}} \overline{|M|^2}$$

P+S p. 107
My notes 57/12:

In this case
$$v_p - v_{p'} = v_{e^-} - v_{e^+} = \frac{p^z}{E} - \frac{p'^z}{E'} = \frac{E}{E} - \frac{(-E)}{E} = 2$$

So we have

$$\left.\frac{d\sigma}{d\Omega}\right|_{cm} = \frac{1}{4E^2}\,\frac{1}{2}\,\frac{1}{16\pi^2}\,\frac{\sqrt{E^2 - M_{(\mu)}^2}}{2E}\, g^4 \left[1 + \frac{M_{(\mu)}^2}{E^2} + \left(1 - \frac{M_{(\mu)}^2}{E^2}\right)\cos^2\theta\right]$$

$$= \frac{\alpha^2}{16E^2}\sqrt{1 - \frac{M_{(\mu)}^2}{E^2}}\left[1 + \frac{M_{(\mu)}^2}{E^2} + \left(1 - \frac{M_{(\mu)}^2}{E^2}\right)\cos^2\theta\right]$$

We can consider two limits

- high-energy $\frac{M_{(\mu)}^2}{E^2} \ll 1$ $\qquad \frac{d\sigma}{d\Omega} \simeq \frac{\alpha^2}{16E^2}\left(1 + \cos^2\theta\right)$
- threshold $\frac{M_{(\mu)}^2}{E^2} \simeq 1$ $\qquad \frac{d\sigma}{d\Omega} \simeq \frac{\alpha^2}{16M_{(\mu)}^2}\,\frac{|\bar{k}|}{M_{(\mu)}^2}$

Q: what happens for $\frac{M_{(\mu)}^2}{E^2} \gg 1$

N.B. See P+S p.137-140 for more information and for an introduction to $e^+e^- \to q\bar{q}$ (experimentally very important!).

Crossing Symmetry

$\boxed{\text{P+S 5.3}}$

Schwartz 13.4

We have been considering $e^+e^- \to \mu^+\mu^-$

But we can actually flip this diagram on its sides and consider instead $e^-\mu^- \to e^-\mu^-$

electron-muon Møller scattering