Scattering in QED

What is all the fuss about anyway? Why have we done all this work just to get here: calculating scattering amplitudes in QED?

QED occupies a special place in the hearts and history of particle physics – it's really where all of the foundations of 20th century particle physics were laid. QED describes the interactions of charged fermions with photons (i.e. EM in QFT language) and is one of the four fundamental forces. It has been spectacularly successful and its success largely drove the efforts to formulate the Standard Model, by applying similar ideas to QCD and the weak force. For example, the anomalous magnetic moment of the electron $a_e = \frac{g-2}{2}$ has been calculated to 10^{th} order in perturbation theory and the theory result is $1\,159\,652\,182.032(720)\times 10^{-12}$. The most precise experimental result is $1\,159\,652\,180.73(28)\times 10^{-12}$!

Laporta has spent ~30 years calculating the 8^{th} order mass independent terms to 1100 digits...

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In 1712.06060 they found an error in an integral in the mass-independent 10^{th} order contribution (there are 12672 Feynman diagrams that are mass independent).

~ 9 equivalent diagrams

Physically, QED incorporates three "generations" of leptons

- electrons $\quad m_e \sim 0.51$ MeV $\quad \rightarrow$ all have spin $1/2$, charge $\pm e$
- muons $\quad m_\mu \sim 106$ MeV
- tauons $\quad m_\tau \sim 1777$ MeV

N.B. neutrinos are leptons, but they are neutral!

To account for these generations, we can modify $\mathcal{L}_{QED}$

$$\mathcal{L}_{QED} = \sum \bar{\psi}_\ell (i\not{\partial} - m_\ell)\psi_\ell - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Here ℓ is a new index (not a spinor index!) representing the type of fermion/lepton - also called its "flavour".

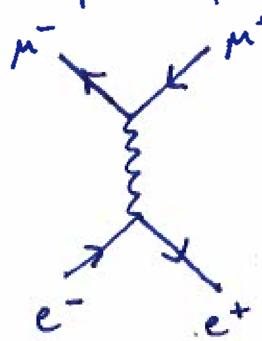
The fact that all leptons listed have the same charge leads to lepton universality - they all interact with photons in the same way. And if the masses were all the same there would be a further symmetry similar to SU(2) isospin in QCD.

The existence of different lepton flavours leads to a multitude of processes - life would be way more boring if all we had were electrons and positrons.

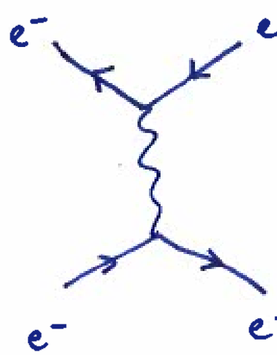
[P+S chap. 5]

Examples

- lepton pair production, e.g. $e^+e^- \rightarrow \mu^+\mu^-$



• Bhabha scattering : $e^+e^- \to e^+e^-$



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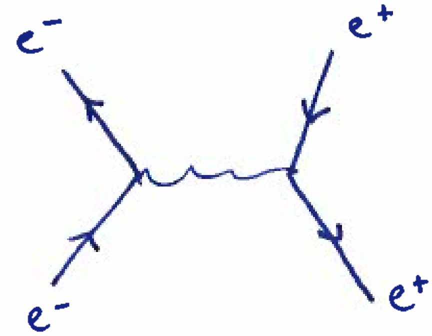
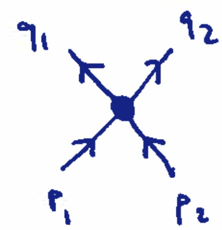


"s-chanel" "t-channel" ← "Mandelstam variables"

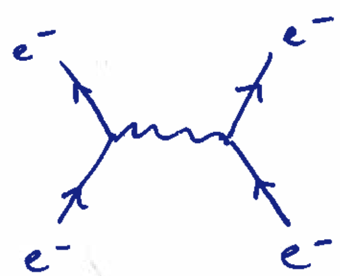


$s = (p_1 + p_2)^2$

$t = (q_1 - p_1)^2$

$u = (q_1 - p_2)^2$

• Møller scattering : $e^-e^- \to e^-e^-$



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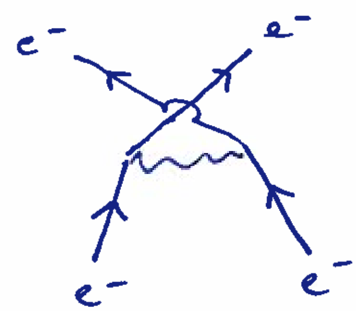
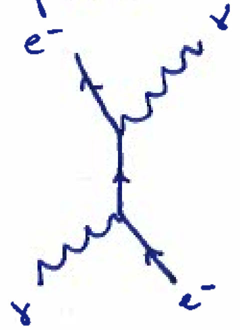


"u-channel"

• Compton scattering : $e^-\gamma \to e^-\gamma$



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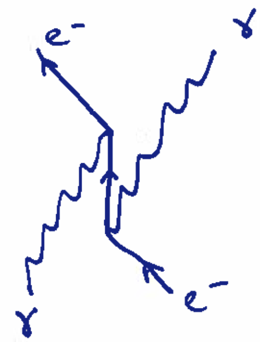
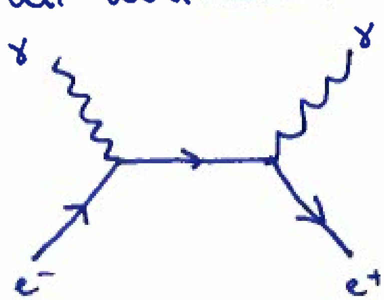


• Pair annihilation : $e^+e^- \to 2\gamma$



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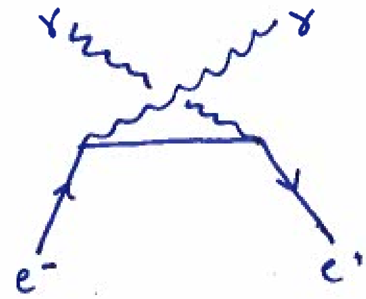
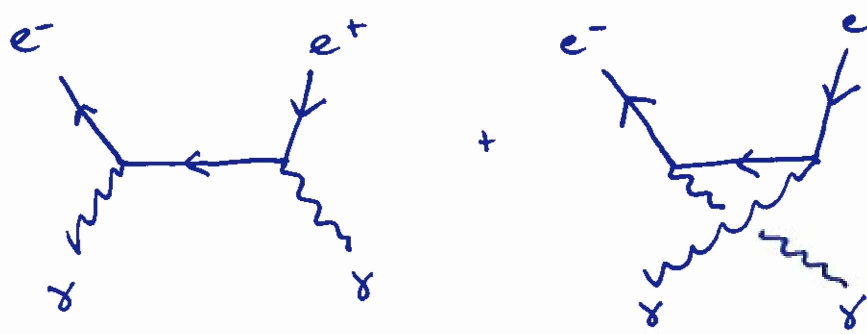
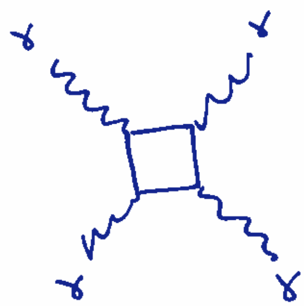


- "Breit-Wheeler process" : $\gamma\gamma' \to e^+e^-$

Predicted 1934
Yet to be observed
in a laboratory!



- Photon scattering : $\gamma\gamma' \to \gamma\gamma'$



← a "box diagram"

__Example : unpolarised $e^+e^- \to \mu^+\mu^-$__

We will start our study of scattering with the first of these, muon pair production. Specifically, we start by calculating the unpolarised cross-section. $\boxed{\text{P+S } 5.1}$

Recall that the cross-section for two final state particles is

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{1}{2E_A E_B |v_A - v_B|} \frac{|\bar{p}_1|}{(2\pi)^2 4E_{cm}} |\mathcal{M}(p_A, p_B \to p_1, p_2)|^2$$

the __invariant matrix element__

obtained from the T-matrix as

$$\langle p_1 \dots p_n | iT | p_A, p_B \rangle = (2\pi)^4 \delta^{(4)}\left(p_A + p_B - \sum_f p_f\right) i\mathcal{M}(p_A, p_B \to \{p_f\})$$

To get the cross-section our procedure is basically

1. write down the Feynman diagrams

2. use these to calculate the invariant matrix element, $i\mathcal{M}$

3. calculate $|\mathcal{M}|^2$

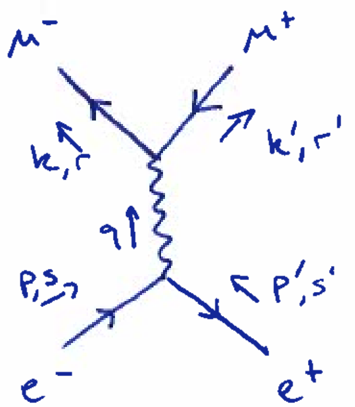
4. use a convenient reference frame to simplify $|\mathcal{M}|^2$

5. calculate the cross-section

So let's just jump right in.

we chose this process partly because there's only one.

$\boxed{1.}$ We know the Feynman diagram



$\boxed{2.}$

$$= \bar{v}^{s'}(\bar{p}')(-ig\gamma^\mu)u^s(\bar{p})\left(\frac{-ig_{\mu\nu}}{q^2}\right)$$

$$\times\ \bar{u}^r(\bar{k})(-ig\gamma^\nu)v^{r'}(k')$$

$$= \underbrace{ig^2\frac{1}{(p+p')^2}\bar{v}^{s'}(\bar{p}')\gamma^\mu u^s(\bar{p})\cdot\bar{u}^r(\bar{k})\gamma_\mu v^{r'}(\bar{k}')}_{=\ i\mathcal{M}}$$

N.B. $s = (p+p')^2$

$\boxed{3.}$ So how do we calculate $|\mathcal{M}|^2$? Clearly we need $\mathcal{M}^*$

$$(i\mathcal{M})^* = -ig^2\frac{1}{(p+p')^2}\left(\bar{v}^{s'}(\bar{p}')\gamma^\mu u^s(\bar{p})\right)^*\left(\bar{u}^r(\bar{k})\gamma_\mu v^{r'}(\bar{k}')\right)^*$$

where

$$(\bar{v}^{s'}(\bar{p}')\gamma^\mu u^s(\bar{p}))^* = (v^\dagger\gamma^0\gamma^\mu u)^* \overset{((\bar{v}\gamma^\mu u)^T)^\dagger}{=} u^\dagger\gamma^{\mu\dagger}\gamma^0 v = u^\dagger\gamma^0\gamma^0\gamma^\mu\gamma^0\gamma^0 v = \bar{u}\gamma^\mu v$$

$$\Rightarrow (i\mathcal{M})^\dagger = -ig^2\frac{1}{(p+p')^2}\bar{u}^s(\bar{p})\gamma^\mu v^{s'}(\bar{p}')\cdot\bar{v}^{r'}(\bar{k}')\gamma_\mu u^r(\bar{k})$$

so

$$|\mathcal{M}|^2 = \frac{g^4}{(p+p')^4}\bar{u}^s(\bar{p})\gamma^\mu v^{s'}(p')\cdot\bar{v}^{s'}(\bar{p}')\gamma^\nu u_s(\bar{p})\cdot\bar{v}^{r'}(\bar{k}')\gamma_\mu u^r(\bar{k})\cdot\bar{u}^r(\bar{k})\gamma_\nu v^{r'}(\bar{k}'$$

The simplest quantity we can calculate is the unpolarised cross-section – extracted from unpolarised beams in an experiment where we don't measure the polarisation of the final state fermions – which means we:

- average over initial state spins
- sum over final state spins

$$\Rightarrow \overline{|M|^2} = \left(\frac{1}{2}\sum_s\right)\left(\frac{1}{2}\sum_{s'}\right)\sum_r\sum_{r'}\left|M(ss' \to rr')\right|^2$$

$$= \frac{1}{4}\frac{g^4}{(p+p')^4}\sum_{ss'} \bar{u}^s(\bar{p})\gamma^\mu v^{s'}(\bar{p}')\,\bar{v}^{s'}(\bar{p})\gamma^\nu u^s(\bar{p})$$

$$\times \sum_{rr'} \bar{v}^{r'}(\bar{k}')\gamma_\mu u^r(\bar{k})\,\bar{u}^r(\bar{k})\gamma_\nu v^{r'}(\bar{k}')$$

Recall our outer product identities! They finally come in useful

$$\sum_{s'} v^{s'}_\beta(\bar{p}')\bar{v}^{s'}_\gamma(\bar{p}') = (\not{p}'-m_e)_{\beta\gamma} \qquad \sum_r u^r_\beta(\bar{k})\bar{u}^r_\gamma(\bar{k}) = (\not{k}+m_\mu)_{\beta\gamma}$$

$$\sum_s u^s_\delta(\bar{p})\bar{u}^s_\alpha(\bar{p}) = (\not{p}+m_e)_{\delta\alpha} \qquad \sum_{r'} \bar{v}^{r'}_\delta(\bar{k}')\bar{v}^{r'}_\alpha(\bar{k}') = (\not{k}'-m_\mu)_{\delta\alpha}$$

$$\Rightarrow \overline{|M|^2} = \frac{1}{4}\frac{g^4}{(p+p')^4}\left[(\not{p}+m_e)^{\delta\alpha}\,\gamma^\mu_{\alpha\beta}\,(\not{p}'-m_e)^{\beta\gamma}\,\gamma^\nu_{\gamma\delta}\right]$$

$$\times \left[(\not{k}'-m_\mu)_{\delta'\alpha'}\,\gamma_\mu^{\alpha'\beta'}\,(\not{k}+m)_{\beta'\gamma'}\,\gamma_\nu^{\gamma'\delta'}\right]$$

$$= \frac{1}{4}\frac{g^4}{(p+p')^4}\,\mathrm{Tr}\left[(\not{p}+m_e)\gamma^\mu(\not{p}'-m_e)\gamma^\nu\right]$$

$$\times\,\mathrm{Tr}\left[(\not{k}'-m_\mu)\gamma_\mu(\not{k}+m)\gamma_\nu\right]$$