Coupling matter and gauge fields

Schwartz 8.3

We have now seen how to relate scattering observables, such as cross-sections, to objects we can calculate in QFT (time-ordered correlation functions), via the LSZ reduction formula, through the S-matrix and Wick's theorem, to Feynman diagrams and Feynman rules.

In this final section of the course, we will apply what we have learned to quantum electrodynamics (QED), the quantum gauge theory of the electromagnetic force.

But first, we have to figure out how to couple matter fields (e.g. electrons and positrons in QED) to gauge fields (photons in QED).

Following Schwartz, however, we start with the slightly simpler case of <u>scalar QED</u>, in which we couple a complex scalar field to photons.

If we simply try to jam together fields ← e.g. $A_\mu \phi \partial^\mu \phi$ (Q: why not $A_\mu \phi \phi$?) we quickly run into issues with gauge invariance ← e.g. $A_\mu \phi \partial^\mu \phi \xrightarrow[A^\mu \to A^\mu + \partial^\mu \alpha]{} A_\mu \phi \partial^\mu \phi + \partial_\mu \alpha \phi \partial^\mu \phi$

To deal with this, we have to allow ϕ to transform somehow under a gauge transformation. This is impossible for a single real scalar field, so we need to have multiple degrees of freedom.

such as a complex scalar field

Coupling a complex scalar field to a gauge field gives us scalar QED.

First we note that we already know one transform for ϕ, ϕ^*

$$\left.\begin{aligned} \phi(x) &\to e^{-i\alpha(x)}\phi(x) \\ \phi^*(x) &\to e^{+i\alpha(x)}\phi^*(x) \end{aligned}\right\} \text{ leaves } m^2\phi^*\phi \text{ invariant}$$

can be a function of x!

Next we note that

$$\partial_\mu\phi^*\partial^\mu\phi$$

is not invariant under this transformation.

because $\partial^\mu(e^{-i\alpha(x)}\phi) = -i(\partial^\mu\alpha)\phi e^{-i\alpha} + e^{-i\alpha}\partial^\mu\phi$

But introducing the covariant derivative

$$D^\mu = \partial^\mu + iA^\mu$$

gives us an invariant term

$$(D_\mu\phi)^* D^\mu\phi$$

$$\begin{aligned} D^\mu\phi = (\partial^\mu + iA^\mu)\phi &\to (\partial^\mu + iA^\mu + i(\partial^\mu\alpha))e^{-i\alpha}\phi \\ &= e^{-i\alpha}\partial^\mu\phi - i(\partial^\mu\alpha)e^{-i\alpha}\phi \\ &\quad + iA^\mu e^{-i\alpha}\phi + i(\partial^\mu\alpha)e^{-i\alpha}\phi \\ &= e^{-i\alpha}(\partial^\mu + iA^\mu)\phi \\ &= e^{-i\alpha}D^\mu\phi \end{aligned}$$

Therefore, the Lagrangian of scalar QED

$$\boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^* D^\mu \phi - m^2 \phi^* \phi}$$

is invariant under the gauge transformation

$$A^\mu \to A^\mu + \frac{1}{e} \partial^\mu \alpha \qquad \phi \to e^{-i\alpha} \phi \qquad \phi^* \to e^{i\alpha} \phi^*$$

- factor e is convention – the covariant derivative changes to
  $D^\mu = \partial^\mu + ie A^\mu$
  – characterises the strength of the coupling between ϕ, ϕ^*, A^μ
- replacing ∂^μ with D^μ is the <u>principle of minimal coupling</u>

<u>Feynman rules for scalar QED</u> $\boxed{\text{Schwartz 9.2}}$

To derive the Feynman rules, let's expand our action

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \phi^* (\partial^2 + m^2) \phi - ie A_\mu \left[\phi^* \partial^\mu \phi - (\partial^\mu \phi^*) \phi \right]$$
$$+ e^2 A_\mu A^\mu |\phi|^2$$

and then we can read off the rules we need

 $= \dfrac{i}{p^2 - m^2 + i\epsilon}$ scalar propagator

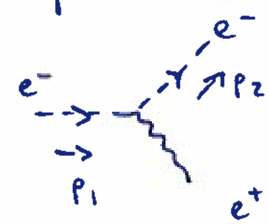
 $= \dfrac{-i}{p^2 + i\epsilon} \left(g_{\mu\nu} - (1-\xi) \dfrac{p_\mu p_\nu}{p^2} \right)$ photon propagator

Let's now look at the interaction terms, starting with the pieces that have derivatives

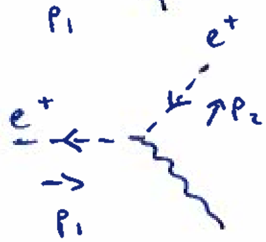
$$-ieA_\mu \left[\phi^* \partial^\mu \phi - (\partial^\mu \phi^*) \phi \right]$$

Note

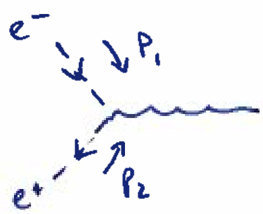
- always one ϕ^*, one ϕ and one A^μ
- derivatives give factors of $\pm ip^\mu$ ← $\phi = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_{\vec{p}} e^{-ip\cdot x} + b^\dagger_{\vec{p}} e^{ip\cdot x} \right)$
- overall factor is $(-ie) \cdot \underset{\substack{\uparrow \\ \text{derivative}}}{i} \cdot \underset{\substack{\uparrow \\ e^{i\mathcal{L}_{int}}}}{i} = ie$
- define $a_{\vec{p}}$ to annihilate "e^-" and $b_{\vec{p}}$ to annihilate "e^+"

⇒ 

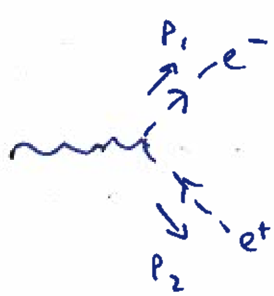
$= ie\left(-p^1_\mu - p^2_\mu\right)$



$= ie\left(p^1_\mu + p^2_\mu\right)$



$= ie\left(-p^1_\mu + p^2_\mu\right)$

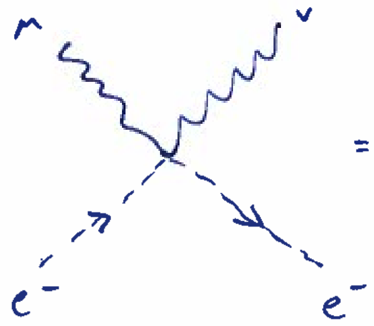


$= ie\left(-p^1_\mu + p^2_\mu\right)$

"ie times the sum of momenta aligned with their particle flow arrows minus the sum of momenta opposite to their particle flow"

N.B. uncharged particles don't care about momentum orientation

Final rule comes from $e^2 A_\mu^* A^\mu |\phi|^2 = e^2 A_\mu A_\nu |\phi|^2 g^{\mu\nu}$

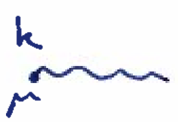


$= 2ie^2 g_{\mu\nu}$

factor of two comes from symmetry of A_μ and A_ν

Wea also need the final states

$= 1$



$= \epsilon^\mu(k)$

$= \epsilon^{*\mu}(k)$

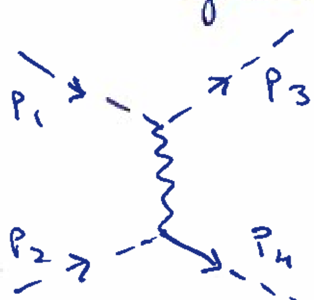
} come from $\overbrace{A^\mu |\bar{p}}\rangle = \epsilon^\mu(p)$
$\langle \overbrace{\bar{p}| A^\mu} = \epsilon^{*\mu}(p)$

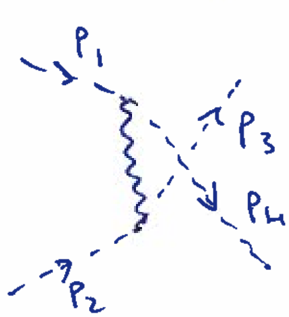
Scattering in scalar QED

$\boxed{\text{Schwartz 9.3}}$

Now that we have our Feynman rules we can apply them to some $2 \to 2$ scattering examples. The first is scalar Møller scattering "e^-e^-" $\Rightarrow$ "e^-e^-". There are two diagrams that we can draw

this is scalar QED so it isn't really $e^-e^- \to e^-e^-$, which is described by QED





The first is in the "t-channel" and the second in the "u-channel", terminology that comes from the Mandelstam variables.

These variables characterise $2 \to 2$ scattering (though they can be generalised)

Schwartz 7.4.1
Tong 3.5.1
P+S 5.4

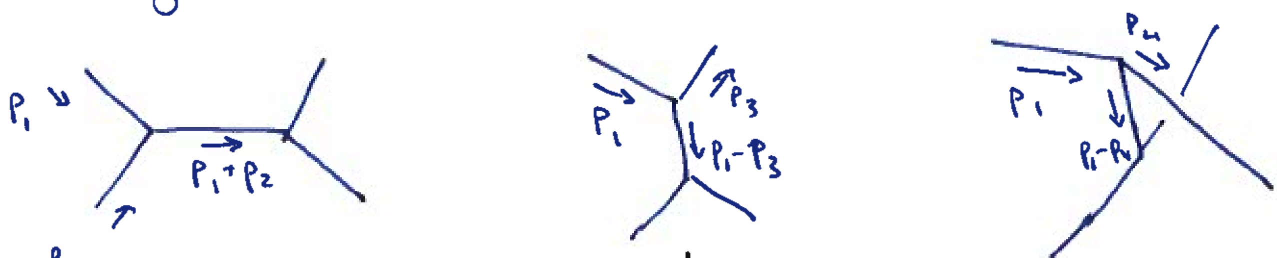
$$s \equiv (p_1 + p_2)^2 = (p_3 + p_4)^2$$

$$t \equiv (p_1 - p_3)^2 = (p_2 - p_4)^2$$

$$u \equiv (p_1 - p_4)^2 = (p_2 - p_3)^2$$

Lorentz invariants!

The channels are defined by the momentum flowing through the intermediate state. In scalar ϕ^3 theory, for example, they are



s — annihilation, $s \geq 0$

t — scattering, $t \leq 0$

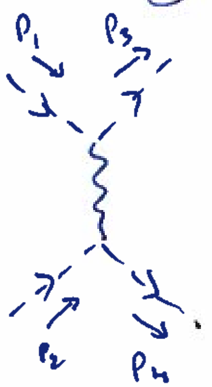
u — also scattering, $u \leq 0$

Note that

$$s + t + u = \sum_{j=1}^{4} m_j^2$$

for particles of mass m_1, m_2, m_3, m_4.

Returning to our scalar Møller scattering, we have



$$= i\mathcal{M}_t = (-ie)(p_1^\mu + p_3^\mu)\left[\frac{-i}{k^2}\left(g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)\right]$$
$$\times(-ie)(p_2^\nu + p_4^\nu)$$

$$k^\mu = p_3^\mu - p_1^\mu$$

To simplify this, we note that

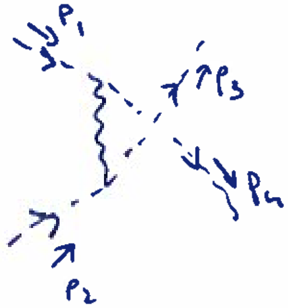
$$k_\mu(p_1^\mu + p_3^\mu) = (p_3^\mu - p_1^\mu)(p_{1\mu} + p_{3\mu})$$
$$= p_3^2 - p_1^2 = m^2 - m^2 = 0$$

so we have

$$i\mathcal{M}_t = i\frac{e^2}{t}(p_1 + p_3)\cdot(p_2 + p_4) \qquad t = k^2 = (p_3 - p_1)^2$$

no ξ dependence! physical scattering is gauge independent

The other contribution is



$$= i\mathcal{M}_u = (-ie)(p_1^\mu + p_4^\mu)\left[\frac{-i}{k^2}\left(g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)\right]$$
$$\times(-ie)(p_2^\nu + p_3^\nu)$$

$$k^\mu = p_4^\mu - p_1^\mu$$

This time we have

$$k_\mu(p_1^\mu + p_4^\mu) = p_4^2 - p_1^2 = m^2 - m^2 = 0$$

Thus

$$iM_{\mu\mu} = \frac{ie^2}{u}\left(p_1^\mu + p_4^\mu\right)\left(p_{2\mu} + p_{3\mu}\right)$$

Putting these together, the cross-section for scalar Møller scattering is

$$\frac{d\sigma}{d\Omega}(\text{"}e^-e^-\text{"} \to \text{"}e^-e^-\text{"}) = \frac{e^4}{64\pi^2 E_{cm}}|M|^2$$

$$= \frac{e^4}{64\pi^2 E_{cm}}\left[\frac{(p_1+p_3)\cdot(p_2+p_4)}{t} + \frac{(p_1+p_4)\cdot(p_2+p_3)}{u}\right]^2$$

$$= \frac{\alpha^2}{4s}\left[\frac{s-u}{t} + \frac{s-t}{u}\right]^2$$

$\alpha = \frac{e^2}{4\pi}$ is the scalar "fine structure constant"

The Ward identity

$\boxed{\text{Schwartz } 9.4}$

The Ward identity is a manifestation of gauge invariance and required

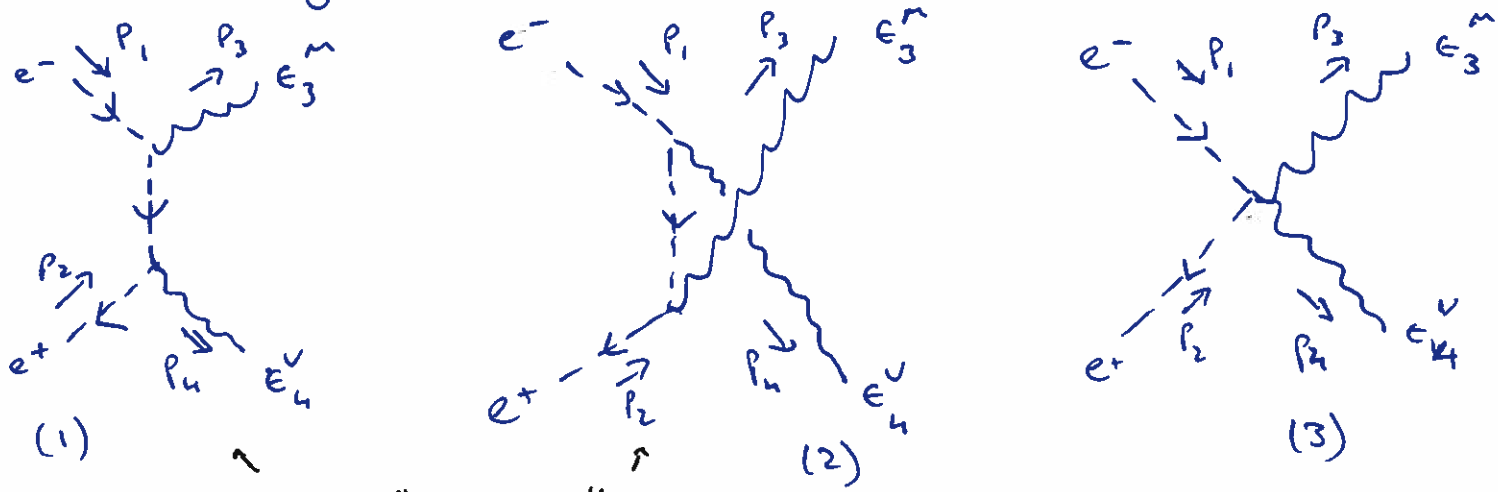
$$p_\mu M^\mu = 0$$

for any matrix element M^μ that involves an onshell photon.

ie has 2 indices

Since the photon propagator is $\Pi_{\mu\nu}$ and the matrix element is a scalar, we must be able to write any matrix element involving a photon propagator in the form $M_{\mu\nu}\Pi^{\mu\nu}$.

An example of a "physical" process for which we can check the Ward identity is "e^+e^-" $\to \gamma\gamma$. The three diagrams that contribute to this are



Q: what "channels" are these two in?

Diagram (1) is

$$iM_t = \frac{(-ie)^2\, i\,(2p_1^\mu - p_3^\mu)(p_4^\nu - 2p_2^\nu)\,\epsilon^*_{3\mu}\epsilon^*_{4\nu}}{(p_1-p_3)^2 - m^2}$$

$$= ie^2 \frac{(p_3\cdot\epsilon_3^* - 2p_1\cdot\epsilon_3^*)(p_4\cdot\epsilon_4^* - 2p_2\cdot\epsilon_4^*)}{p_3^2 - 2p_3\cdot p_1}$$

$p_1^2 - m^2 = 0$ because external "e^-" is onshell

Diagram (2) is

$$iM_u = ie^2 \frac{(p_3\cdot\epsilon_3^* - 2p_2\cdot\epsilon_3^*)(p_4\cdot\epsilon_4^* - 2p_1\cdot\epsilon_4^*)}{p_3^2 - 2p_3\cdot p_1}$$

can get this by switching $p_3 \leftrightarrow p_4$ (or $p_1 \leftrightarrow p_2$)

The sum of these two does not satisfy the Ward identity (obviously they don't alone)

$$\mathcal{M}_t + \mathcal{M}_u \underset{\uparrow}{=} e^2 \left[p_4 \cdot \epsilon_4^* - 2p_2 \cdot \epsilon_4^* + p_4 \cdot \epsilon_4^* - 2p_1 \cdot \epsilon_4^* \right]$$

replace $\epsilon_3^{*\mu}$ with p_3^μ

$$= 2e^2 \epsilon_4^{*\mu} \left(p_{4\mu} - p_{2\mu} - p_{1\mu} \right)$$

But when we add in the third diagram

$$i\mathcal{M}_4 = 2ie^2 g_{\mu\nu} \epsilon_3^{*\mu} \epsilon_4^{*\nu}$$

and then replace $\epsilon_3^{*\mu}$ with p_3^μ we have

$$i\mathcal{M}_4 \underset{\epsilon_3^{*\mu} \to p_3^\mu}{=} 2ie^2 p_3 \cdot \epsilon_4^*$$

So the sum of the three diagrams is

$$\mathcal{M}_t + \mathcal{M}_u + \mathcal{M}_4 = 2e^2 \epsilon_4^{*\mu} \left(p_{4\mu} - p_{2\mu} - p_{1\mu} + p_{3\mu} \right)$$

$$= 0$$

because conservation of 4-momentum requires $p_1 + p_2 = p_3 + p_4$!