We have just said that our choice of $1/4!$ was dictated by the X vertex. Beyond leading order, this comes from the four-point function. Let's look at [ie λ^0] that in more detail.

$$\langle\Omega|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|\Omega\rangle = \frac{\langle 0|T\{\phi_1\phi_2\phi_3\phi_4 e^{-i\int dt\, H_I(t)}\}|0\rangle}{\langle 0|T\{e^{-i\int dt\, H_I(t)}\}|0\rangle}$$

$$= \langle 0|T\{\phi_1\phi_2\phi_3\phi_4\}|0\rangle$$

$$+ (-i)\frac{\lambda}{4!}\int d^4z\, \langle 0|T\{\phi_1\phi_2\phi_3\phi_4\, \phi_z\phi_z\phi_z\phi_z\}|0\rangle$$

$$+ \frac{1}{2!}(-i)^2\frac{\lambda^2}{(4!)^2}\int d^4z_1\, d^4z_2\, \langle 0|T\{\phi_1\phi_2\phi_3\phi_4\, \phi_{z1}\phi_{z1}\phi_{z1}\phi_{z1} \times \phi_{z2}\phi_{z2}\phi_{z2}\phi_{z2}\}|0\rangle$$

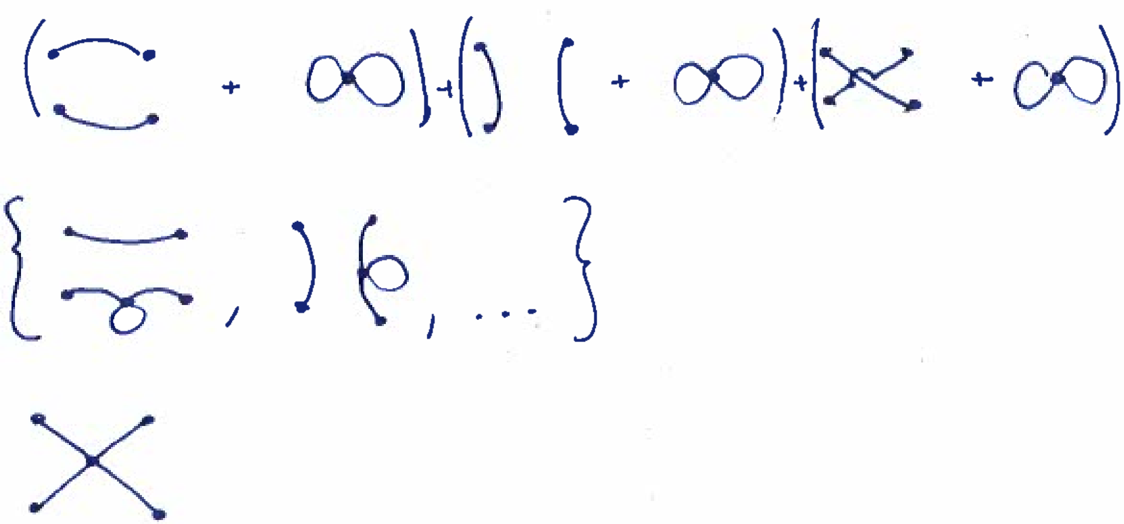
$$+ O(\lambda^3)$$

Order-by-order this is

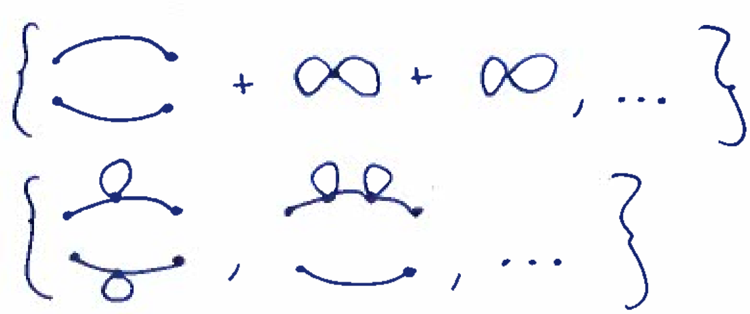
← Recall last lecture

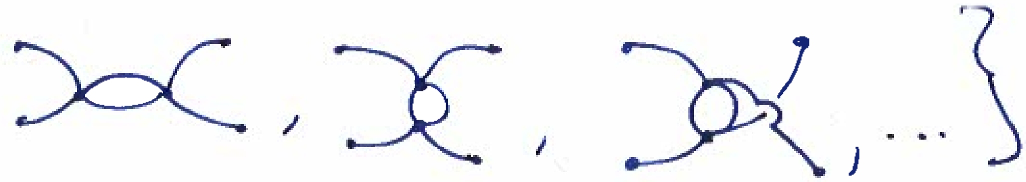
- $O(\lambda^0)$ =

- $O(\lambda^1)$ =



- $O(\lambda^2)$ =



$+ \Big\{$  $\Big\}$

"connected" and "undressed"

Also: "one-particle irreducible" – cutting one propagator does not cause diagram to fall apart.

connected – all external vertices are continuously connected to all other external vertices

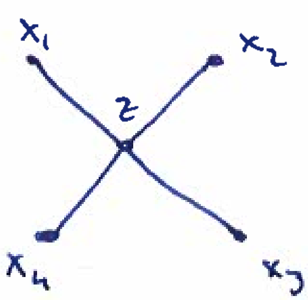
undressed – all propagators are just simple propagators (with none that look like )

So far we have worked in position space, but in most cases it is much easier to work in momentum space. For that we need the momentum space Feynman rules.

$$D_F(x-y) = i\int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon}$$

$\Rightarrow$  $= \tilde{D}_F(p) = \dfrac{i}{p^2-m^2+i\epsilon}$

Vertex is given by



$$= -i\lambda \int d^4z\, D_F(x_1-z)\, D_F(x_2-z)\, D_F(x_3-z)\, D_F(x_4-z)$$

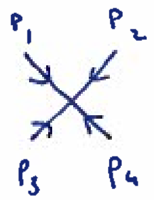
$$= -i\lambda \int d^4z \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} \int \frac{d^4p_3}{(2\pi)^4} \int \frac{d^4p_4}{(2\pi)^4}$$

$$\times \frac{i e^{-ip_1\cdot(x_1-z)}}{p_1^2-m^2+i\epsilon} \cdot \frac{i e^{-ip_2\cdot(x_2-z)}}{p_2^2-m^2+i\epsilon} \cdot \frac{i e^{-ip_3\cdot(x_3-z)}}{p_3^2-m^2+i\epsilon} \cdot \frac{i e^{-ip_4\cdot(x_4-z)}}{p_4^2-m^2+i\epsilon}$$

$$= -i\lambda \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} \int \frac{d^4p_3}{(2\pi)^4} \int \frac{d^4p_4}{(2\pi)^4} \; \frac{i e^{-ip_1\cdot x_1}}{p_1^2-m^2+i\epsilon} \frac{i e^{-ip_2\cdot x_2}}{p_2^2-m^2+i\epsilon} \frac{i e^{-ip_3\cdot x_3}}{p_3^2-m^2+i\epsilon} \frac{i e^{-ip_4\cdot x_4}}{p_4^2-m^2+i\epsilon}$$

$$\times \underbrace{\int d^4z \; e^{i(p_1+p_2+p_3+p_4)\cdot z}}_{(2\pi)^4 \delta^{(4)}(p_1+p_2+p_3+p_4)}$$

$(2\pi)^4 \delta^{(4)}(p_1+p_2+p_3+p_4)$ <=> represents momentum conservation at the vertex

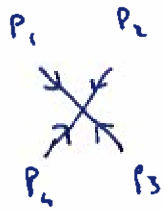
$\Rightarrow$ 

$= -i\lambda (2\pi)^4 \delta^{(4)}(p_1+p_2+p_3+p_4)$

So our Feynman rules are: $\boxed{\text{Schwartz 7.3}}$

- propagator 

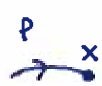
$= \dfrac{i}{p^2-m^2+i\epsilon}$

- vertex 

$= -i\lambda (2\pi)^4 \delta^{(4)}(p_1+p_2+p_3+p_4)$

↑ direction is somewhat arbitrary but fixed by delta function


$= -i\lambda (2\pi)^4 \delta^{(4)}(p_1-p_2+p_3+p_4)$

- external point 

$= e^{-ip\cdot x}$

↑ N.B. 

$= e^{+ip\cdot x}$

- integrate over undetermined momenta

- divide by symmetry factor

We're looking good – but there's an important piece we've forgotten – the denominator !

$\boxed{P+S\ 4.3}$

Recall our formula was

$$\langle\Omega|\phi^H(x_1)\,\phi^H(x_2)\ldots\phi^H(x_n)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\{\phi_1\phi_2\cdots\phi_n\, e^{-i\int_{-T}^{T}d\tau\, H_I(\tau)}\}|0\rangle}{\langle 0|T\{e^{-i\int_{-T}^{T}d\tau\, H_I(\tau)}\}|0\rangle}$$

What happens when we expand this denominator ?

$$\langle 0|T\{e^{-i\int d\tau H_I(\tau)}\}|0\rangle = \langle 0|T\{1 + \frac{(-i\lambda)}{4!}\int d^4z\,\phi^4(z) + O(\lambda^2)\}|0\rangle$$

$$= \langle 0|0\rangle + \frac{(-i\lambda)}{4!}\int d^4z\,\langle 0|\phi^4(z)|0\rangle + O(\lambda^2)$$

$$= 1 + \frac{(-i\lambda)}{4!}\,3\cdot\int d^4z\ 8 + O(\lambda^2)$$

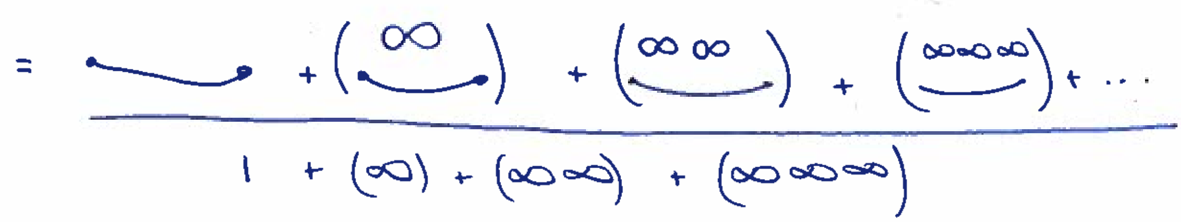
But we've seen such <u>disconnected</u> terms before – in the numerator!

A careful analysis (see p. 95–98 of Peskin + Schroeder) shows that all such disconnected diagrams cancel.

Rather than prove this in generality, let's look at our favorite example

We interpret these as "vacuum fluctuations" – the interacting vacuum is not "empty" !

$$\langle\Omega|T\{\phi^H(x)\phi^H(y)\}|\Omega\rangle = \frac{\langle 0|T\{\phi(x)\phi(y)\,e^{-i\int d\tau H_I(\tau)}\}|0\rangle}{\langle 0|T\{e^{-i\int d\tau H_I(\tau)}\}|0\rangle}$$



$$= \frac{D_F(x-y)\left(1 + \infty + \frac{1}{2}(\infty)^2 + \frac{1}{3!}(\infty)^3 + \ldots\right)}{\left(1+\infty+\frac{1}{2}(\infty)^2 + \frac{1}{3!}(\infty)^3+\ldots\right)}$$

$$= \frac{D_F(x-y)\exp(\infty)}{\exp(\infty)}$$

$$= D_F(x-y)$$

Finally, then our rule becomes

$$\boxed{\begin{aligned}&\langle\Omega|T\{\phi(x_1)\ldots\phi(x_n)\}|\Omega\rangle \\ &\qquad = \text{sum over all connected diagrams} \\ &\qquad\quad \text{with } n \text{ external points } \{x_1, x_2, \ldots, x_n\}\end{aligned}}$$

connected = all external points continuously connected to all others

Scattering Example

Schwartz 7.4

Calculate the leading order contributions to the process

$$\phi\phi \to \phi\phi$$

in ϕ^3 theory

$$\mathcal{L} = -\frac{1}{2}\phi\partial^2\phi - \frac{1}{2}m^2\phi^2 + \frac{g}{3!}\phi^3$$

The cross-section in the centre of mass frame is

$$\frac{d\sigma}{d\Omega}(\phi\phi \to \phi\phi) = \frac{1}{64\pi^2 E_{cm}^2}|\mathcal{M}|^2$$

Step 1.

Write down Feynman diagrams

Step 2

Use these to calculate $i\mathcal{M}$

Step 3

calculate $|\mathcal{M}|^2$

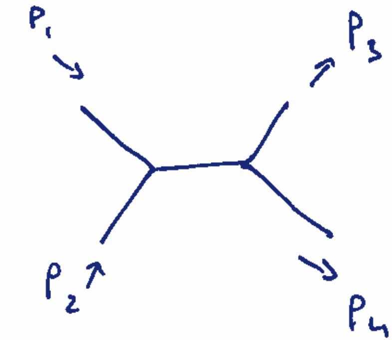
Step 4

use a convenient reference frame to simplify $|\mathcal{M}|^2$

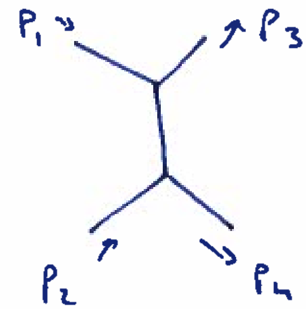
Step 1

Diagrams are

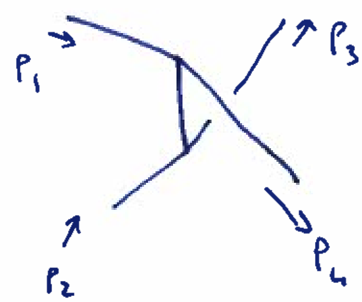
(1) =



(2) =



(3) =



Step 2

Invariant matrix elements are

$$i\mathcal{M}_1 = (ig)\frac{i}{(p_1+p_2)^2 - m^2 + i\epsilon}(ig) = \frac{-ig^2}{s^2 - m^2 + i\epsilon}$$ ← "s-channel"

$$i\mathcal{M}_2 = (ig)\frac{i}{(p_1-p_3)^2 - m^2 + i\epsilon}(ig) = \frac{-ig^2}{t - m^2 + i\epsilon}$$ ← "t-channel"

$$i\mathcal{M}_3 = (ig)\frac{i}{(p_1-p_4)^2 - m^2 + i\epsilon}(ig) = \frac{-ig^2}{u - m^2 + i\epsilon}$$ ← "u-channel"

Step 3

$$|\mathcal{M}|^2 = |i\mathcal{M}_1 + i\mathcal{M}_2 + i\mathcal{M}_3|^2$$

$$= g^4 \left[\frac{1}{s-m^2+i\epsilon} + \frac{1}{t-m^2+i\epsilon} + \frac{1}{u-m^2+i\epsilon}\right]^2$$

Step 4 ← In this case CM is a convenient frame

$$\frac{d\sigma}{d\Omega}(\phi\phi \to \phi\phi) = \frac{g^4}{64\pi^2 E_{cm}} \left[\frac{1}{s-m^2+i\epsilon} + \frac{1}{t-m^2} + \frac{1}{u-m^2}\right]^2$$

And that's it! (At leading order...)

Q: What about next-to-leading order (i.e. g^4 contributions to $i\mathcal{M}$)