Wick's theorem

$\boxed{\text{Schwartz 7.A}}$ P+S 4.3, long 5.5

We have reduced our problem to calculating objects like

$$\langle 0 | T\{\phi_I(x)\,\phi_I(x_2)\ldots\phi(x_n)\}|0\rangle$$

Q: Why didn't I include H_I here?

This is great! We already know what the answer is for $n=2$

$$\langle 0|T\{\phi_I(x_1)\,\phi_I(x_2)\}|0\rangle = \langle 0|T\{\phi^{(\text{FREE})}(x_1)\,\phi^{(\text{FREE})}(x_2)\}|0\rangle$$

$$= D_F(x_1 - x_2)$$

the Feynman propagator!

What about $n>2$?

Well we know how to write the interaction picture field in terms of the annihilation and creation operators of the free theory. So we could just plug those in and work it all out. But there's an easier way...

To understand this procedure, let's look again at the $n=2$ case.

Recall
$$\phi_I(x) = \int \frac{d^3\bar{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a(\bar{p})e^{-ip\cdot x} + a^\dagger(\bar{p})e^{ip\cdot x}\right)$$

$$= \underbrace{\int \frac{d^3\bar{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}a(\bar{p})e^{-ip\cdot x}}_{\phi_I^+(x)} + \underbrace{\int \frac{d^3\bar{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}a^\dagger(\bar{p})e^{ip\cdot x}}_{\phi_I^-(x)}$$

If we take $x^0 > y^0$ then

$$T\{\phi_I(x)\phi_I(y)\} = \phi_I(x)\phi_I(y)$$
$$= (\phi_I^+(x) + \phi_I^-(x))(\phi_I^+(y) + \phi_I^-(y))$$
$$= \phi_I^+(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^+(y) + \phi_I^+(x)\phi_I^-(y) + \phi_I^-(x)\phi_I^-(y)$$

normal ordering $\rightarrow$
$$= \phi_I^+(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(y)\phi_I^+(x) + [\phi_I^+(x), \phi_I^-(y)] + \phi_I^-(x)\phi_I^-(y)$$

Thus

$$T\{\phi_I(x)\phi_I(y)\} \underset{x^0>y^0}{=} :\phi_I(x)\phi_I(y): + D(x-y)$$

↙ recall $D(x-y) = [\phi(x),\phi(y)]$ in the free theory $= [\phi_I^+(x), \phi_I^-(y)]$

↑ since all terms are normal ordered

If we repeat the argument for $y^0 > x^0$ we have

$$T\{\phi_I(x)\phi_I(y)\} \underset{y^0>x^0}{=} :\phi_I(x)\phi_I(y): + D(y-x)$$

Putting these together, we have

$$T\{\phi_I(x)\phi_I(y)\} = :\phi_I(x)\phi_I(y): + \Delta_F(x-y)$$

↑ Feynman propagator

$$\Delta(x-y) = \theta(x^0-y^0)[\phi_I^+(x), \phi_I^-(y)] + \theta(y^0-x^0)[\phi_I^+(y), \phi_I^-(x)]$$

We can introduce some new notation here

> contraction: replace a pair of fields in a string of fields by the Feynman propagator, leaving all others untouched.

↑ denote by $\overbracket{\qquad}$

<u>Examples</u>

$$\overbracket{\phi_I(x)\phi_I(y)} = \Delta_F(x-y)$$

$$\phi_I(x)\overbracket{\phi_I(y)\phi_I(z)} = \phi_I(x)\Delta_F(x-y)$$

With this new notation we can write

$$T\{\phi_I(x)\phi_I(y)\} = :\phi_I(x)\phi_I(y): + \overbracket{\phi_I(x)\phi_I(y)}$$

$$= :\phi_I(x)\phi_I(y) + \overbracket{\phi_I(x)\phi_I(y)}:$$

↲ We can bring "inside" normal ordering symbol because Δ_F is just a c-number.

$T\{\phi(x)\phi(y)\}$ and $:\phi(x)\phi(y):$ are <u>operators</u> but the difference between them is a c-number!

This is our first example of <u>Wick's Theorem</u>

| Proof is by induction <u>P+S, p. 90</u>

Wick's Theorem: For any string of fields

$$T\{\phi_I(x_1)\phi_I(x_2)\cdots\phi_I(x_n)\} = :\phi_I(x_1)\phi_I(x_2)\cdots\phi_I(x_n) + \text{all possible pair contractions}:$$

This is particularly useful, because we (generally) want to calculate vacuum matrix elements and for those the normal-ordered term will vanish.

$\boxed{\text{Schwartz 7.2.6}}$
P+S 4.4

<u>Example</u>

$$\langle 0|T\{\phi_I(x)\phi_I(y)\}|0\rangle = \langle 0|\overset{?\,0}{\cancel{:\phi_I(x)\phi_I(y):}}|0\rangle + \langle 0|\Delta_F(x-y)|0\rangle$$
$$= \Delta_F(x-y)\langle 0|0\rangle$$
$$= \Delta_F(x-y)$$

<u>Example</u>

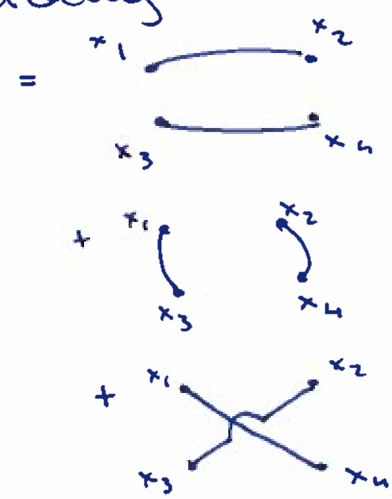
$$\begin{aligned}
T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\} = \; :&\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4) \\
&+ \wick{\c1\phi_I(x_1)\c1\phi_I(x_2)}\phi_I(x_3)\phi_I(x_4) \\
&+ \wick{\c1\phi_I(x_1)\phi_I(x_2)\c1\phi_I(x_3)}\phi_I(x_4) \\
&+ \wick{\c1\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\c1\phi_I(x_4)} \\
&+ \phi_I(x_1)\wick{\c1\phi_I(x_2)\c1\phi_I(x_3)}\phi_I(x_4) \\
&+ \phi_I(x_1)\wick{\c1\phi_I(x_2)\phi_I(x_3)\c1\phi_I(x_4)} \\
&+ \phi_I(x_1)\phi_I(x_2)\wick{\c1\phi_I(x_3)\c1\phi_I(x_4)} \\
&+ \wick{\c1\phi_I(x_1)\c1\phi_I(x_2)\c2\phi_I(x_3)\c2\phi_I(x_4)} \\
&+ \wick{\c1\phi_I(x_1)\c2\phi_I(x_2)\c1\phi_I(x_3)\c2\phi_I(x_4)} \\
&+ \wick{\c1\phi_I(x_1)\c2\phi_I(x_2)\c2\phi_I(x_3)\c1\phi_I(x_4)}:
\end{aligned}$$

$$= :\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4): + \Delta_F(x_1-x_2):\phi_I(x_3)\phi_I(x_4): + \Delta_F(x_1-x_3):\phi_I(x_2)\phi_I(x_4):$$
$$+ \Delta_F(x_1-x_4):\phi_I(x_2)\phi_I(x_3): + \Delta_F(x_2-x_3):\phi_I(x_1)\phi_I(x_4):$$
$$+ \Delta_F(x_2-x_4):\phi_I(x_1)\phi_I(x_3): + \Delta_F(x_3-x_4):\phi_I(x_1)\phi_I(x_2):$$
$$+ \Delta_F(x_1-x_2)\Delta_F(x_3-x_4) + \Delta_F(x_1-x_3)\Delta_F(x_2-x_4)$$
$$+ \Delta_F(x_1-x_4)\Delta_F(x_2-x_3)$$

So the vacuum matrix element is just

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\}|0\rangle = \Delta_F(x_1-x_2)\Delta_F(x_3-x_4) + \Delta_F(x_1-x_3)\Delta_F(x_2-x_4) + \Delta_F(x_1-x_4)\Delta_F(x_2-x_3)$$

which we can also represent graphically



Wick's theorem is exactly what we need to evaluate arbitrary correlation functions of the form

$$\langle 0|T\{\phi_1\phi_2\ldots\phi_n\, e^{-i\int_{-T}^{T} d\tau\, H_I(\tau)}\}|0\rangle$$

$$\phi_i \equiv \phi(x_i)$$

To do this, we expand the exponential and use Wick's theorem. Let's just take $n=2$ for now. Then

$$\langle 0|T\{\phi_I(x)\phi_I(y)e^{-i\int dt\, H_I(\tau)}\}|0\rangle$$

$$= \underbrace{\langle 0|T\{\phi_I(x)\phi_I(y)\}|0\rangle}_{D_F(x-y)} + (-i)\langle 0|T\{\phi_I(x)\phi_I(y)\int dt\, H_I(\tau)\}|0\rangle + \ldots$$

$$= D_F(x-y) - i\frac{\lambda}{4!}\langle 0|T\{\phi_I(x)\phi_I(y)\int d^4z\, \phi_I(z)\phi_I(z)\phi_I(z)\phi_I(z)\}|0\rangle + O(\lambda^2)$$

$$\uparrow$$

$$\int dt\, H_I = \int dt \int d^3\bar{z}\, \mathcal{H}_I = \int d^4z\, \mathcal{H}_I$$

Let's look at the second term ($O(\lambda)$). We keep all pairwise contractions.

$\Rightarrow$ if $\underbrace{\phi_I(x)\phi_I(y)}_{= D_F(x-y)}$ $\times$ $\left\{\begin{array}{l} \phi_I(z)\phi_I(z)\;\phi_I(z)\phi_I(z) \\ \phi_I(z)\phi_I(z)\;\phi_I(z)\phi_I(z) \\ \phi_I(z)\phi_I(z)\;\phi_I(z)\phi_I(z) \end{array}\right\}$ = 3 identical objects $D_F(z-z)D_F(z-z)$

$\Rightarrow$ if $\phi_I(x)\,\phi_I(y)\,\phi_I(z)\,\phi_I(z)\,\phi_I(z)\,\phi_I(z) \rightarrow \phi_y\phi_z\phi_z\phi_z,\ \phi_y\phi_z\phi_z\phi_z,\ \phi_y\phi_z\phi_z\phi_z$

$\phi_I(x)\,\phi_I(y)\,\phi_I(z)\,\phi_I(z)\,\phi_I(z)\,\phi_I(z) \rightarrow$ 3 more options

$\phi_I(x)\,\phi_I(y)\,\phi_I(z)\,\phi_I(z)\,\phi_I(z)\,\phi_I(z) \rightarrow$ 3 more options

$\phi_I(x)\,\phi_I(y)\,\phi_I(z)\,\phi_I(z)\,\phi_I(z)\,\phi_I(z) \rightarrow$ 3 more options

12 identical objects

$D_F(x-z)D_F(y-z)D_F(z-z)$

This is all the possibilities!

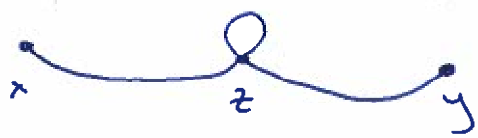
So we can write

$$\langle 0 | T\{\phi_I(x)\,\phi_I(y)\,(-i)\frac{\lambda}{4!}\int d^4z\,\phi_I^4(z)\}|0\rangle$$

$$= 3\cdot(-i)\frac{\lambda}{4!}\,D_F(x-y)\int d^4z\,D_F(z-z)\,D_F(z-z)$$

$$+\,12\cdot(-i)\frac{\lambda}{4!}\int d^4z\,D_F(x-z)\,D_F(y-z)\,D_F(z-z)$$

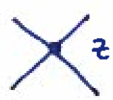
$\simeq$ 

$+$ 

In fact we can use this diagrammatic notation to express and correlation function. Our example of $\lambda\phi^4$ theory has two graphical elements:

propagators 

$= D_F(x-y)$

vertices 

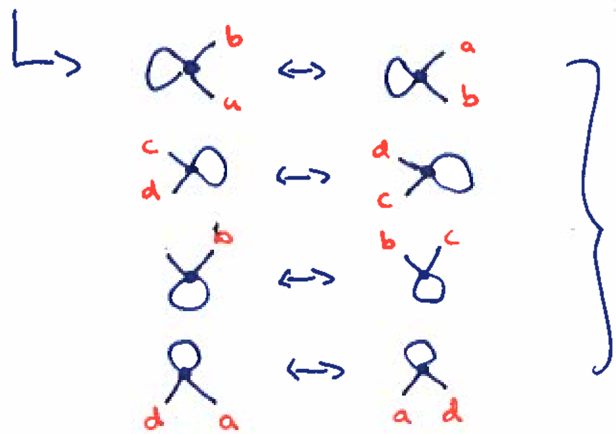
$= -i\lambda\int d^4z$ Q: why no factor of 4! ?

and symmetry factors

$\boxed{\text{Symmetry factor: divide by the number of permutations of internal elements that leave the diagram unchanged}}$

In the example we considered we have two diagrams

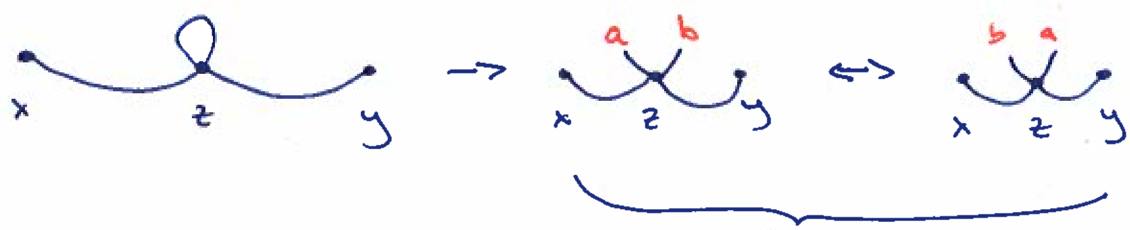
no symmetry factor



8 possibilities

$\Rightarrow$ symmetry factor is $1/8$

N.B $\frac{3}{4!} = 1/8$ ✓



2 possibilities $\Rightarrow$ symmetry factor $= 1/2$

N.B. $\frac{12}{4!} = \frac{1}{2}$ ✓

Now we see why we chose a factor of $4!$ - it simplifies our symmetry factors and our position space Feynman rule is just $(-i\lambda)$.

Q: What happens with ϕ^3 theory?