Interaction picture

Aim of all of this interaction picture stuff: show that

$$\langle \Omega | T\{\phi(x_1)\phi(x_2)\cdots\phi(x_n)\}|\Omega\rangle = \frac{\langle 0|T\{\phi_I(x_1)\cdots\phi_I(x_n)e^{-i\int dt\, H_I}\}|0\rangle}{\langle 0|T\{e^{-i\int dt\, H_I}\}|0\rangle}$$

↑ eventually used in scattering amplitudes

Recall:

- Schrödinger picture – states evolve – operators don't $\qquad i\frac{d}{dt}|\psi\rangle_s = H|\psi\rangle_s$

- Heisenberg picture – operators evolve – states don't $\qquad O_H(t) = e^{iHt}O_s e^{-iHt}$, $\quad |\psi\rangle_H = e^{iHt}|\psi\rangle_s$

Introduce:

- Interaction picture – hybrid of the two
  – split $H = H_0 + H_{INT}$

  H_0: controls time dependence of operators; H_{INT}: controls time dependence of states

Split into $H_0 + H_{INT}$ is arbitrary but useful if H_0 is exactly solvable

$$\Rightarrow |\psi\rangle_I = e^{iH_0 t}|\psi\rangle_s$$
$$O_I(t) = e^{iH_0 t}O_s e^{-iH_0 t}$$

P+S 4.2

Tong 3.1

this applies to H_{INT} so we have

$$H_I \equiv (H_{INT})_I = e^{iH_0 t}(H_{INT})_s e^{-iH_0 t}$$

then the Schrödinger equation in the interaction picture is

$$i\frac{d}{dt}|\psi\rangle_s = H_s|\psi\rangle_s \Rightarrow i\frac{d}{dt}\left(e^{-iH_0 t}|\psi\rangle_I\right) = (H_0 + H_{INT})_s e^{-iH_0 t}|\psi\rangle_I$$

chain rule

$$\Rightarrow i\frac{d}{dt}|\psi\rangle_I = e^{iH_0 t}(H_{INT})_s e^{-iH_0 t}|\psi\rangle_I = H_I|\psi\rangle_I$$

Dyson's formula

Schwartz 7.2

This discussion has been formulated in quantum mechanics, but of course we are interested in a field theoretic formulation.

To keep things at least somewhat concrete, we will have in mind the $\lambda\phi^4$ theory, so that

$$H = H_0 + H_{INT} = \underbrace{\int d^3\bar{x}\left(\frac{1}{2}\pi^2 + \frac{1}{2}\bar{\nabla}\phi\cdot\bar{\nabla}\phi + \frac{m^2}{2}\phi^2\right)} + \overbrace{\frac{\lambda}{4!}\int d^3\bar{x}\,\phi^4}^{H_{INT}}$$

The time dependence of fields is given by ← "Heisenberg" field

$$\phi(\bar{x},t) = e^{iHt}\,\phi(\bar{x},0)\,e^{-iHt}$$

$$\Rightarrow \phi(\bar{x},t) = e^{iHt}\,e^{-iH_0t}\,\underbrace{e^{iH_0t}\,\phi(\bar{x},0)\,e^{-iH_0t}}\,e^{iH_0t}\,e^{-iHt}$$

$$\phi_I(\bar{x},t) = e^{iH_0t}\,\phi(\bar{x},0)\,e^{-iH_0t}$$

For $\lambda \ll 1$, this gives the dominant time evolution in the theory → time evolution in the free theory!

$$\Rightarrow \phi(\bar{x},t) = U_I^{\dagger}(t,0)\,\phi_I(\bar{x},t)\,U_I(t,0)$$

$$U_I(t,0) \equiv e^{iH_0t}\,e^{-iHt}$$

The interaction picture field satisfies (by definition)

$$\phi_I(\bar{x},t) = e^{iH_0t}\,\phi(\bar{x},0)\,e^{-iH_0t}$$

but clearly $\phi_I(\bar{x},0) = \phi(\bar{x},0)$ so we have

$$\phi_I(\bar{x},t) = e^{iH_0t}\,\phi_I(\bar{x},0)\,e^{-iH_0t}$$

Thus the interaction field obeys the same time evolution as the free-field $\Rightarrow$ we can use our previous decomposition

$$\phi_I(x) = \int \frac{d^3\bar{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(e^{-ip\cdot x}\,a(\bar{p}) + e^{ip\cdot x}\,a^{\dagger}(\bar{p})\right)$$

Now the interaction Hamiltonian, in the interaction picture, is

$$H_I(t) \equiv e^{iH_0t}\,(H-H_0)\,e^{-iH_0t}$$

$$= e^{iH_0t}\,(H_{INT})\,e^{-iH_0t}$$

make it concrete →

$$= e^{iH_0t}\left(\frac{\lambda}{4!}\int d^3\bar{x}\,\phi^4\right)e^{-iH_0t}$$

$$= \frac{\lambda}{4!}\int d^3\bar{x}\; e^{iH_0t}\phi\, e^{-iH_0t}\, e^{iH_0t}\phi\, e^{-iH_0t}\, e^{iH_0t}\phi\, e^{-iH_0t}\, e^{iH_0t}\phi\, e^{-iH_0t}$$

$$= \frac{\lambda}{4!}\int d^3\bar{x}\;\phi_I^4$$

So... we have written the time dependence of the "Heisenberg" field in terms of a new "interaction" field and a time evolution operator. Now we need to express that operator in terms of the interaction field.

To do this, we note that

$$i \frac{\partial}{\partial t} U(t,t') = i \frac{\partial}{\partial t} \left(e^{iH_0 t} e^{-iH(t-t')} e^{-iH_0 t'} \right)$$

$$= e^{iH_0 t} (-H_0) e^{-iH(t-t')} e^{-iH_0 t'} + e^{iH_0 t} H e^{-iH(t-t')} e^{-iH_0 t'}$$

$$= e^{iH_0 t} (H - H_0) e^{-iH(t-t')} e^{-iH_0 t'}$$

$$= e^{iH_0 t} (H - H_0) e^{-iH_0 t} e^{iH_0 t} e^{-iH(t-t')} e^{-iH_0 t'}$$

$$= H_I(t)\, U(t,t')$$

The <u>formal</u> solution to this is given by — Dyson's series/formula

$$\boxed{U(t,t') = T\left\{ \exp\left(-i \int_{t'}^{t} d\tau\, H_I(\tau) \right) \right\}} \qquad \text{for } U(t,t) = 1$$

Before we show this, let's first show how the naive solution

$$U(t,t') \stackrel{?}{=} \exp\left(-i \int_{t'}^{t} d\tau\, H_I(\tau) \right) \quad \text{fails.}$$

First we have to clarify what this exponential means

$$\exp\left(-i \int_{t'}^{t} d\tau\, H \right) = 1 - i \int_{t'}^{t} H_I\, d\tau + \frac{(-i)^2}{2} \left(\int_{t'}^{t} H_I\, d\tau \right)^2 + \ldots$$

Then apply $\partial/\partial t$:

$$\frac{\partial}{\partial t} \exp\left(-i \int_{t'}^{t} d\tau\, H \right) = -i H_I(t) - \frac{1}{2} \left(\int_{t'}^{t} H_I\, d\tau \right) H_I(t) - \frac{1}{2} H_I(t) \left(\int_{t'}^{t} H_I\, d\tau \right)$$

not good, wrong ordering and we can't commute

looks good $\sim H_I U$

So this naive solution does not work – and this is where the time ordering comes in.

Recall $T\{O_1(t_1)O_2(t_2)\} = \begin{cases} O_1(t_1)O_2(t_2) & t_1 > t_2 \\ O_2(t_2)O_1(t_1) & t_2 > t_1 \end{cases}$

Let's now show that the time-ordered exponential <u>does</u> work.

This is the "Dyson series" approach.

First let's choose, without loss of generality $t' = 0$ and write a formal solution by iteration

$$U_I(t,0) = 1 + (-i)\int_0^t d\tau \, H_I(\tau) U_I(\tau, 0)$$

$$\Rightarrow U_I(t,0) = 1 + (-i)\int_0^t d\tau_1 \, H_I(\tau_1)\left(1 + (-i)\int_0^{\tau_1} d\tau_2 \, H_I(\tau_2) U_I(\tau_2, 0)\right)$$

$$= 1 + (-i)\int_0^t d\tau \, H_I(\tau) + (-i)^2 \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \, H_I(\tau_1) H_I(\tau_2) U_I(\tau_2, 0)$$

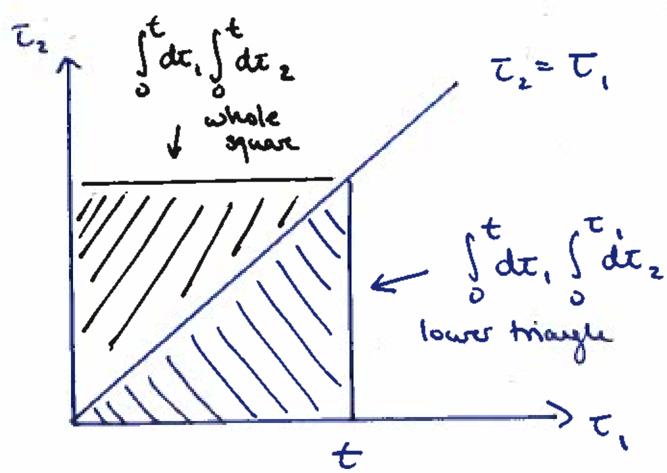
keep repeating this →

$$\vdots$$

$$= 1 + (-i)\int_0^t d\tau \, H_I(\tau)$$
$$+ (-i)^2 \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \, H_I(\tau_1) H_I(\tau_2)$$
$$+ (-i)^3 \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 \, H_1(\tau_1) H_I(\tau_2) H_I(\tau_3)$$
$$+ \dots$$

Now we notice that

$$\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \, H_I(\tau_1) H_I(\tau_2) = \frac{1}{2}\int_0^t d\tau_1 \int_0^t d\tau_2 \, T\{H_I(\tau_1) H_I(\tau_2)\}$$



Proof :

$$\int_0^t d\tau_1 \int_0^t d\tau_2 \; T\{H_I(\tau_1) H_I(\tau_2)\} = \underbrace{\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2}_{\tau_1 > \tau_2} H_I(\tau_1) H_I(\tau_2)$$

$$+ \int_0^t d\tau_2 \overbrace{\int_0^{\tau_2} d\tau_1}^{\tau_2 > \tau_1} H_I(\tau_2) H_I(\tau_1) \quad \text{relabel } \tau_1 \leftrightarrow \tau_2$$

$$= \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \; H_I(\tau_1) H_I(\tau_2) + \int_0^t d\tau_1 \int_0^{\tau} d\tau_2 H_I(\tau_1) H_I(\tau_2)$$

$$= 2 \int_0^t d\tau_1 \int_0^{\tau_1} d\tau \; H_I(\tau_1) H_I(\tau_2)$$

#

In fact one can use the same sort of arguments to show

$$\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{n-1}} d\tau_n \; H_I(\tau_1) H_I(\tau_2) \ldots H_I(\tau_n)$$

$$= \frac{1}{n!} \int_0^t d\tau_1 \int_0^t d\tau_2 \ldots \int_0^t d\tau_n \; T\{H_I(\tau_1) H_I(\tau_2) \ldots H_I(\tau_n)\}$$

So our series solution becomes

$$U_I(t,0) = 1 + (-i) \int_0^t d\tau_1 T\{H_I(\tau_1)\}$$

$$+ \frac{(-i)^2}{2!} \int_0^t d\tau_1 \int_0^t d\tau_2 \; T\{H_I(\tau_1) H_I(\tau_2)\}$$

$$+ \frac{(-i)^3}{3!} \int_0^t d\tau_1 \int_0^t d\tau_2 \int_0^t d\tau_3 \; T\{H_I(\tau_1) H_I(\tau_2) H_I(\tau_3)\}$$

$$+$$

$$\equiv T\left\{e^{-i \int_0^t d\tau \, H_I(\tau)}\right\} \qquad \left[\overset{\text{Recall}}{=} e^{iH_0 t} e^{-iHt}\right]$$

#

Now we have a (formal) solution for our time-evolution operator, let's look at some of its properties.

First : $U_I(t,t') = T\left\{\exp\left(-i \int_{t'}^t d\tau \, H_I(\tau)\right)\right\}$

$= e^{iH_0 t} e^{-iH(t-t')} e^{-iH_0 t'}$

satisfies : $i \frac{\partial}{\partial t} U_I(t,t') = H_I(t) U_I(t,t')$

} collected here for completeness

Then we also have

$$i \frac{\partial}{\partial t'} U(t,t') = - U_I(t,t') H_I(t')$$

To show this we first note

$$U_I^\dagger(t,t') = e^{iH_0 t'} e^{iH(t-t')} e^{-iH_0 t} = e^{iH_0 t'} e^{-iH(t'-t)} e^{-iH_0 t} = U_I(t',t)$$

then we start from

$$i \frac{\partial}{\partial t} U_I(t,t') = H_I(t) U_I(t,t')$$

and relabel $t \Leftrightarrow t'$

$$i \frac{\partial}{\partial t'} U_I(t',t) = H_I(t') U_I(t',t)$$

$$i \frac{\partial}{\partial t'} U_I^\dagger(t,t') = H_I(t') U_I^\dagger(t,t')$$

and dagger this whole equation

$$- i \frac{\partial}{\partial t'} U_I(t,t') = U_I(t,t') H_I(t') \quad \#$$

↓ we use the fact that the Hamiltonian is Hermitian (what happens if it is not ?!).

N.B. It <u>is</u> possible to have a non-Hermitian Hamiltonian if your system is PT-symmetric. Unfortunately our Universe is not. See e.g. hep-th/0303005

We can show that time translations work just as we expect

$$U_I(t_1,t_2) U_I(t_2,t_3) = e^{iH_0 t_1} e^{iH(t_1-t_2)} e^{-iH_0 t_2} \cdot e^{iH_0 t_2} e^{iH(t_2-t_3)} e^{-iH_0 t_3}$$

$$= e^{iH_0 t_1} e^{iH(t_1-t_3)} e^{-iH_0 t_3}$$

$$= U_I(t_1,t_3)$$

and

$$U_I(t_1,t_3) U_I^\dagger(t_2,t_3) = U_I(t_1,t_3) U_I(t_3,t_2) = U_I(t_1,t_2).$$

We have now achieved the first of our aims - we have written the Heisenberg field entirely in terms of the interaction field (which, recall, propagates like a free field!).

So far, so good, but also, so what? Well, we're trying to understand correlation functions in the interacting theory and for that we need one more piece - the vacuum state.

The vacuum in the interacting theory, $|\Omega\rangle$, is not the vacuum state in the free theory, $|0\rangle$! ie $|\Omega\rangle \neq |0\rangle$!

First, $|\Omega\rangle$ is an eigenstate of H, specifically the ground state

$$H|\Omega\rangle = E_\Omega |\Omega\rangle \qquad E_\Omega < E_n \quad \forall\, n>0$$

note $E_\Omega \neq 0$, which is defined by $H_0|0\rangle = 0$

$$H|n_{INT}\rangle = E_n |n_{INT}\rangle$$

take $\langle\Omega|\Omega\rangle = 1$

To isolate this eigenstate, imagine starting with $|0\rangle$ and assume $\langle\Omega|0\rangle \neq 0$ ← otherwise H_{INT} is not a small perturbation!

$$\begin{aligned} e^{-iHT}|0\rangle &= \sum_n e^{-iE_nT}|n\rangle\langle n|0\rangle \\ &= e^{-iE_\Omega T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-iE_nT}|n\rangle\langle n|0\rangle \end{aligned}$$

Since $E_n > E_\Omega \;\; \forall\, n>0$, the first term will dominate as $T \to \infty(1-i$

$$\Rightarrow |\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_\Omega T}\langle\Omega|0\rangle\right)^{-1} e^{-iHT}|0\rangle$$

comes from $T\to\infty(1-i\epsilon)$ for $e^{-iHT}|0\rangle = e^{-iE_\Omega T}\langle\Omega|0\rangle|$

Similarly we can write

$$\langle\Omega| = \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_\Omega T}\langle 0|\Omega\rangle\right)^{-1}\langle 0|e^{-iHT}$$

↑
N.B. same limit

We can write these in a more useful form. Since T is huge, we can shift is by a small constant, t_0

$$|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_\Omega(T+t_0)}\langle\Omega|0\rangle\right)^{-1} e^{-iH(T+t_0)}|0\rangle$$

$$= \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_\Omega(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1} e^{-iH(t_0-(-T))}\underbrace{e^{-iH_0(-T-t_0)}|0\rangle}$$

$H_0|0\rangle = 0$
$\Rightarrow e^{-iH_0 t}|0\rangle = |0\rangle$

$$= \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_\Omega(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1} U_I(t_0,-T)|0\rangle$$

Similarly

$$\langle\Omega| = \lim_{T\to\infty(1-i\epsilon)} \langle 0|U(T,t_0)\left(e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle\right)^{-1}$$

So, up to a multiplicative complex constant, we can obtain the interacting theory vacuum state by time evolving the free theory vacuum state!

We are now in a position to start studying correlation functions

Let's make it concrete by considering $\lambda\phi^4$ theory and the two point function $\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle$.

To make our lives simpler, take $x^0 > y^0 > t_0$

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_\Omega(T-t_0)}\langle 0|\Omega\rangle\right)^{-1}\langle 0|U_I(T,t_0)$$
$$\times \left(U_I(x^0,t_0)\right)^{\dagger}\phi_I(x)\,U(x^0,t_0)$$
$$\times \left(U_I(y^0,t_0)\right)^{\dagger}\phi_I(y)\,U(y^0,t_0)$$
$$\times\, U_I(t_0,-T)|0\rangle\left(e^{-iE_\Omega(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1}$$

$$= \lim_{T\to\infty(1-i\epsilon)} \left(|\langle 0|\Omega\rangle|^2 e^{-2iE_\Omega T}\right)^{-1}$$
$$\times \langle 0|U(T,x^0)\,\phi_I(x)\,U_I(x^0,y^0)\,\phi_I(y)\,U_I(y^0,-T)|0\rangle$$

This looks pretty promising, except for that pesky factor out front. We can deal with that by using $\langle\Omega|\Omega\rangle = 1$

$$\Rightarrow \quad 1 = \langle\Omega|\Omega\rangle$$

$$= \lim_{T\to\infty(1-i\epsilon)} \langle 0|U_I(T,t_0)\left(e^{-iE_\Omega(T-t_0)}\langle 0|\Omega\rangle\right)^{-1}$$

$$\times \left(e^{-iE_\Omega(t_0+T)}\langle\Omega|0\rangle\right)^{-1} U_I(t_0,-T)|0\rangle$$

$$= \lim_{T\to\infty(1-i\epsilon)} \left(|\langle 0|\Omega\rangle|^2 e^{-i2E_\Omega T}\right)^{-1} \langle 0|U_I(T,t_0)U_I(t_0,-T)|0\rangle$$

Multiplying our expression by unity in this form gives us

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle \underset{x^0>y^0}{=} \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|U_I(T,x^0)\,\phi_I(x)\,U_I(x^0,y^0)\,\phi_I(y)U_I(y^0,-T)|0\rangle}{\langle 0|U(T,-T)|0\rangle}$$

This equation also holds if $y^0 > x^0$, because the fields on both sides are time ordered.

$$\Rightarrow \langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \langle 0|U(T,-T)|0\rangle^{-1}$$

$$\times \langle 0|T\{U_I(T,x^0)\,\phi_I(x)\,U_I(x^0,y^0)\,\phi_I(y)\,U_I(y^0,-T)\}|0\rangle$$

$$= \lim_{T\to\infty(1-i\epsilon)} \langle 0|U(T,-T)|0\rangle^{-1}$$

$$\times \langle 0|T\{\phi_I(x)\,\phi_I(y)\,U_I(T,x^0)\,U_I(x^0,y^0)\,U_I(y^0,-T)\}|0\rangle$$

$$= \lim_{T\to\infty(1-i\epsilon)} \langle 0|U(T,-T)|0\rangle^{-1}$$

$$\times \langle 0|T\{\phi_I(x)\,\phi_I(y)\,U_I(T,-T)\}|0\rangle$$

$$= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\{\phi_I(x)\,\phi_I(y)\,e^{-i\int_{-T}^{T}d\tau\, H_I(\tau)}\}|0\rangle}{\langle 0|T\{e^{-i\int_{-T}^{T}d\tau\, H_I(\tau)}\}|0\rangle}$$

The generalisation to an abitrary number of fields follows the same logic and leads us to

$$\boxed{\langle\Omega|T\{\phi(x_1)\phi(x_2)\dots\phi(x_n)\}|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\dots\phi_I(x_n)\, e^{-i\int_{-T}^{T}d\tau\, H_I(\tau)}\}|0\rangle}{\langle 0|T\{e^{-i\int_{-T}^{T}d\tau\, H_I(\tau)}\}|0\rangle}}$$

That's it! We have written time-ordered correlation functions entirely in terms of correlation functions of (time-ordered) fields in the interaction picture – and we know how those behave: they evolve in time just as the free fields do.

Some comments:

- this expression is exact, but is most useful when H_I is small (ie $\lambda \ll 1$) and we can approximate the exponential well with just a few (or one) term(s).
- $U_I(\infty,-\infty)$ is called the "scattering matrix" or "S-matrix"

Really it should be "scattering operator"

Lots of time was spent in the mid 20th Century trying to solve QFTs through the unitarity of the S-matrix, but this is no longer a focus of most QFT