Spin 1

Schwartz 8.4

Tong 6.2

see page 81

Recall that we quantised the complex scalar field as two real fields with a "polarisation" vector. Quantising a vector field is much the same, except now we have a true polarisation vector in there

Q: Is this a massive or massless field?

$$A_\mu(x) = \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \sum_{j=1}^{3} \left[\epsilon_\mu^j(p)\, a_{p,j}\, e^{-ip\cdot x} + \epsilon_\mu^{j*}(p)\, a_{p,j}^\dagger\, e^{ip\cdot x} \right]$$

sum over polarisations

separate creation and annihilation operators for each polarisation

To define our states, we need not just the momentum, but also the polarisation

$$\Rightarrow \quad a_{p,j}^\dagger |0\rangle = \frac{1}{\sqrt{2\omega_p}} |p, \epsilon^j\rangle$$

Normally we would go ahead and write down equal-time commutation relations and so on, but we have to think a little more carefully, because of our old friend gauge invariance.

Here we will focus on the massless case, but similar considerations apply to the massive spin-1 particle.

Our field is

$$A_\mu(x) = \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \sum_{j=1}^{2} \left[\epsilon_\mu^j(p) a_{p,j} e^{-ip\cdot x} + \epsilon_\mu^{j*}(p) a_{p,j}^+ e^{ip\cdot x} \right]$$

and our conjugate momentum field is

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0 \qquad \leftarrow \text{ recall } \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$\pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = -F^{0i} = E^i$$

π^0 vanishes! Because A^0 is not a dynamical degree of freedom, it is redundant. Carrying on, we can calculate the Hamiltonian

$$H = \int d^3\bar{x} \; \pi^i \dot{A}_i - \mathcal{L}$$

$$= \int d^3\bar{x} \left\{ \frac{1}{2}\left(\bar{E}\cdot\bar{E} + \bar{B}\cdot\bar{B}\right) - A_0(\bar{\nabla}\cdot\bar{E}) \right\}$$

A_0 is here a Lagrange multiplier that imposes Gauss' law $\bar{\nabla}\cdot\bar{E} = 0$.

To quantise this fully and apply commutation relations on our field, we will first pick a gauge.

$\bar{\nabla} \cdot A = 0$

Coulomb gauge

The correct commutation relations are

$$\boxed{[A_i(\bar{x}), E_j(\bar{y})] = i\left(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}\right)\delta^{(3)}(\bar{x}-\bar{y})}$$

To see why the more "obvious" choice

$$[A_i(x), E_j(\bar{y})] = i\delta_{ij}\,\delta^{(3)}(\bar{x}-\bar{y})$$

does not work, consider

$$[\bar{\nabla}\cdot\bar{A}(x), \bar{\nabla}\cdot\bar{E}(\bar{y})] = i\bar{\nabla}^2\delta^{(3)}(\bar{x}-\bar{y}) \neq 0$$

violates $\bar{\nabla}\cdot\bar{A} = \bar{\nabla}\cdot\bar{E} = 0$

The (correct) commutator in momentum space is

$$[A_i(\bar{x}), E_j(\bar{y})] = i\int \frac{d^3\bar{p}}{(2\pi)^3}\left(\delta_{ij} - \frac{p_i p_j}{|\bar{p}|^2}\right) e^{i\bar{p}\cdot(\bar{x}-\bar{y})}$$

and so, for example,

$$[\partial_i A_i(\bar{x}), E_j(\bar{y})] = i\int \frac{d^3\bar{p}}{(2\pi)^3}\left(\delta_{ij} - \frac{p_i p_j}{|\bar{p}|^2}\right) i p_i e^{i\bar{p}\cdot(\bar{x}-\bar{y})}$$

$$= -\int \frac{d^3\bar{p}}{(2\pi)^3}\left(p_j - p_j\frac{|\bar{p}|^2}{|\bar{p}|^2}\right) e^{i\bar{p}\cdot(\bar{x}-\bar{y})}$$

$$= 0$$

as required!

This commutation relation leads to more usual-looking commutation relations for the creation and annihilation operators

← all others zero

$$\boxed{[a_{p,j}, a^{\dagger}_{q,k}] = (2\pi)^3 \delta_{jk} \delta^{(3)}(\bar{p} - \bar{q})}$$

assuming the completeness relation for the polarisation vectors

$$\sum_{r=1}^{2} \epsilon_r^i(\bar{p}) \epsilon_r^j(\bar{p}) = \delta^{ij} - \frac{p^i p^j}{|\bar{p}|^2}$$

In Coulomb gauge our Hamiltonian becomes

$$:H: = \int \frac{d^3\bar{p}}{(2\pi)^3} |\bar{p}| \sum_{j=1}^{2} a^{\dagger}_{p,j} a_{p,j}$$

↖ because the $A_0(\bar{\nabla}\cdot\bar{E})$ term vanishes in Coulomb gauge !

Notes:

- David Tong presents a nice discussion of quantisation in "Lorentz gauge" in his section 6.2.2

  ↖ He means Lorenz – different person !

- Schwartz discusses in more detail why polarisation vectors depend on p^{μ}, and why this relates to the little group of the Lorentz group $(SO(3))$ in his section 8.4.1.

Photon propagator

$\boxed{\text{Schwartz 8.5}}$

The time-ordered two-point function of the vector field is

$$\langle 0|T\{A^\mu(x)A^\nu(y)\}|0\rangle = i\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot(x-y)}\,\Pi^{\mu\nu}(p)$$

this is just a definition of $\Pi^{\mu\nu}$ for now

To find $\Pi^{\mu\nu}$ we would try to calculate the classical Green's function and then apply the $i\epsilon$ prescription. But...

... the classical equation of motion involves a noninvertible operator. Consider the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu$$

external source

which has (classical) equations of motion

$$\partial_\mu F^{\mu\nu} = J^\nu$$

$$\Rightarrow (\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu)A^\mu = J_\nu$$

In momentum space this is

$$(-p^2 g_{\mu\nu} + p_\mu p_\nu)\tilde{A}^\mu = \tilde{J}_\nu$$

which would be solved by $A^\mu = \Pi^{\mu\nu}J_\nu$ if we could invert the operator $(-p^2 g_{\mu\nu} + p_\mu p_\nu)$.

Unfortunately

$$(-p^2 g_{\mu\nu} + p_\mu p_\nu) p^\mu = (-p^2 p_\nu + p^2 p_\nu) = 0$$

so this operator has an eigenvector (p^μ) with a zero eigenvalue

$$\Rightarrow \det(-p^2 g_{\mu\nu} + p_\mu p_\nu) = 0$$

and therefore this operator is noninvertible.

This is a consequence of gauge invariance – A^μ is not uniquely determined by J^μ.

Choosing a gauge is another option, but in practice it is tricky to keep track of the gauge constraint throughout calculations. The simplest trick to do is to add a gauge-fixing term to the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 - J_\mu A^\mu$$

this is an "$\underline{R_\xi}$ gauge" or "covariant gauge"

The equations of motion become

$$\partial_\mu A^\mu = 0 \qquad \leftarrow \text{ from } \xi$$

$$\left[-p^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right) p_\mu p_\nu\right] \tilde{A}^\nu(p) = \tilde{J}_\mu(p)$$

This operator has inverse

$$\Pi^{\mu\nu} = \frac{-p^2 g^{\mu\nu} + (1-\xi) p^\mu p^\nu}{(p^2)^2}$$

$$\hookrightarrow \left[-p^2 g_{\mu\alpha} + \left(1-\tfrac{1}{\xi}\right) p_\mu p_\alpha\right] \frac{1}{(p^2)^2} \left(-p^2 g^{\alpha\nu} + (1-\xi) p^\alpha p^\nu\right)$$

$$= \frac{1}{(p^2)^2} \Big\{ (p^2)^2 g_\mu{}^\nu - p^2\left(1-\tfrac{1}{\xi}\right) p_\mu p^\nu - p^2 (1-\xi) p_\mu p^\nu$$

$$+ \left(1-\tfrac{1}{\xi}\right)(1-\xi) p_\mu p^\nu p^2 \Big\}$$

$$= g_\mu{}^\nu - \frac{1}{p^2}\left[1 - \tfrac{1}{\xi} + 1 - \xi - \frac{(\xi-1)(1-\xi)}{\xi}\right] p_\mu p^\nu$$

$$= g_\mu{}^\nu$$

$$2 - \xi - \tfrac{1}{\xi} - \tfrac{1}{\xi}(\xi - 1 - \xi^2 + \xi)$$

$$= 2 - \xi - \tfrac{1}{\xi} - 1 + \tfrac{1}{\xi} + \xi - 1 = 0 \; !$$

Thus in R_ξ gauges, the time-ordered two-point function is given by

$$\boxed{\langle 0 | T\{A^\mu(x) A^\nu(y)\} | 0 \rangle = \int \frac{d^4 p}{(2\pi)^4} e^{ip\cdot(x-y)} \frac{-i}{p^2 + i\epsilon} \left(g^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}\right)}$$

Common gauges:

- $\xi = 1$ Feynman gauge
- $\xi = 0$ Landau gauge
- $\xi = \infty$ Unitary gauge (not useful for QED)

Other (non covariant) gauges

- $n_\mu A^\mu = 0$ lightcone gauge
- $x_\mu A^\mu = 0$ radial gauge
- $\bar{\nabla} \cdot \bar{A} = 0$ Coulomb gauge

Learning outcomes - scattering

You will be able to:

- identify relevant, marginal and irrelevant couplings
- describe when perturbation theory is applicable
- define cross-sections and decay rates
- relate scattering observables to the S-matrix
- write down the definition of the Lorentz-invariant phase space
- write down expressions for 2-> n differential cross-sections in terms of the matrix element M and for 1-> n decay rates
- write down the LSZ reduction formula
- explain the conceptual significance of the LSZ reduction formula
- define the interaction picture
- state Dyson's formula
- state Wick's theorem
- apply Wick's theorem to scalar fields
- state Feynman rules for ϕ^4 scalar field theory

Interacting (scalar) fields

← We will deal with interacting spinor and vector fields when we get to QED.

So far we've considered free fields - the objects we obtain by applying the quantisation condition

$[\phi(\bar{x},t), \pi(\bar{y},t)] = i\delta^{(3)}(\bar{x}-\bar{y})$ to the Lagrangian $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2)$

We showed that the eigenstates of this theory are states of n "particles", each with energy $E_k = \sqrt{|\bar{k}|^2 + m^2}$. But nothing interesting happens to these eigenstates - to make life interesting, we have to add interactions. These are terms higher order in the fields

$$\rightarrow \quad \mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) - \overbrace{\sum_{n\geq 3}\frac{\lambda_n}{n!}\phi^n}^{\mathcal{L}_{INT}}$$

λ_n are coupling constants

← for convenience (will become apparent later)

Such interaction terms will couple different Fourier modes and allow particles to talk to each other (interact).

What restrictions do we have on possible interactions?

- causality ⇒ put all fields at the same spacetime point, so our theory is "local".
- dimensionality ⇒ note that $[\mathcal{L}] = 4$ ← follows from $e^{i\hbar^{-1}S} = e^{iS}$ ⇒ $[S]=0$

  also $[x] = -1$, $[\partial_\mu] = 1$

  so $[\phi] = 1$

  $[m] = 1$

  $[\lambda_n] = 4-n$

this is a one-page summary of a topic that takes weeks is QFT 2 : effective field theory and the renormalisation group

$\Rightarrow$ three cases:

1) $n < 4$ so $[\lambda_n] > 0$ and the dimensionless combination is $\frac{\lambda_n}{E^{4-n}}$ where E is some mass/energy scale, usually taken to be the scale of the process of interest. Then $\frac{\lambda_n}{E^{4-n}}$ is small at high energies and large at low energies. The coupling is described as <u>relevant</u> because is relevant at low energies, where most stuff happens.

2) $n = 4$ so $[\lambda_n] = 0$ and the coupling is dimensionless. Then $\lambda_n \ll 1$ is the requirement for the application of <u>perturbation theory</u>. Coupling is <u>marginal</u>.

3) $n > 4$ so $[\lambda_n] < 0$ and the dimensionless combination is $\lambda_n E^{n-4}$. This is small at low energies and large at high energies; the coupling is called <u>irrelevant</u>.

- perturbatarity $\Rightarrow$ our theory is only analytically tractable if the coupling is weak, ie $\lambda \ll 1$.

not a real word

in this course we will always work in the "weak coupling regime"

But strongly coupled theories are really cool:
- bound states + confinement
- charge fractionalisation
- dualities w/ gravity

see Tong p. 50

We should be grateful most couplings are irrelevant – it makes QFT simple. But it also makes it hard to work out whats going on in our GUT up at M_{Planck}.

We will discuss three examples of interacting theories

- "ϕ^4 theory" ← treated extensively by Peskin + Schroeder

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2 - \frac{\lambda_4}{4!}\phi^4\right)$$

- Scalar Yukawa theory

treated by Tong

$$\mathcal{L} = \partial_\mu\psi^*\partial^\mu\psi - M_\psi^2\psi^*\psi + \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m_\phi^2\phi^2\right) - g\psi^*\psi\phi$$

complex scalar — real scalar — $g \ll m_\phi, M_\psi$

- Quantum electrodynamics ... eventually ← first we have to understand fermions and gauge symmetri

Let's consider ϕ^4-theory:

- equation of motion now $(\partial^2 + m^2)\phi = -\frac{\lambda}{3!}\phi^3$

can no longer be solved using Fourier decomposition

$\Rightarrow$ to solve this we need new techniques

- "solving" means "diagonalising the Hamiltonian" or finding the energy eigenstates

- We start by first noting there is a new term in the Hamiltonian $H_{INT} = -L_{INT} = \frac{\lambda}{4!}\int d^3\underline{x}\,\phi^4(x)$

Ultimately, what we want to calculate are correlation functions - and finding all n-point functions is another way of "solving" the theory.