Spin 1/2

Schwartz 12.3

Tong 5.2, P+S 3.5

If we try to impose commutation relations on our spinor fields, we rapidly run into trouble, in particular the commutation relations for the creation and annihilation operators pick up a minus sign (which is worse than it sounds) and the Hamiltonian is unbounded from below (which is terrible indeed).

See D. Tong's discussion in his section 5.1

The solution is related to the fact that fermions do not commute, they <u>anticommute</u>!

i.e. they pick up a minus sign when we interchange identical fermions.

In quantum mechanics we think of this as the Pauli exclusion principle. In QFT it stems from the spin-statistics theorem

We impose, instead, equal-time anticommutation relations

$$\boxed{\begin{aligned}\{\psi_\alpha(\bar{x}), \psi^\dagger_\beta(\bar{y})\} &= \delta^{(3)}(\bar{x}-\bar{y})\,\delta_{\alpha\beta}\\ \{\psi_\alpha(\bar{x}), \psi_\beta(\bar{y})\} &= \{\psi^\dagger_\alpha(\bar{x}), \psi^\dagger_\beta(\bar{y})\} = 0\end{aligned}}$$

Writing the fields as

these carry the spinor structure

$$\psi(x) = \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \sum_s \left(a^s_{\bar{p}} u^s(p) e^{-ip\cdot x} + b^{s\dagger}_{\bar{p}} v^s(p) e^{ip\cdot x} \right)$$

$$\bar{\psi}(x) = \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \sum_s \left(b^s_{\bar{p}} \bar{v}^s(p) e^{-ip\cdot x} + a^{\dagger}_{\bar{p}} \bar{u}^s(p) e^{ip\cdot x} \right)$$

we also have

$$\boxed{\begin{aligned} \{a^r_p, a^s_{\bar{q}}\} &= \{b^r_{\bar{p}}, b^s_{\bar{q}}\} = 0 \\ \{a^r_p, a^{s\dagger}_{\bar{q}}\} &= \{b^r_{\bar{p}}, b^{s\dagger}_{\bar{q}}\} = (2\pi)^3 \delta^{(3)}(\bar{p}-\bar{q}) \delta^{rs} \end{aligned}}$$

We define our vacuum to be

$$a^s_{\bar{p}}|0\rangle = b^s_{\bar{p}}|0\rangle = 0$$

one-particle states are

$$|\bar{p}, s\rangle = \sqrt{2\omega_{\bar{p}}}\, a^{s\dagger}_{\bar{p}} |0\rangle$$

$$\langle \bar{p}, r | \bar{q}, s\rangle = 2\omega_{\bar{p}} (2\pi)^3 \delta^{(3)}(\bar{p}-\bar{q}) \delta^{rs}$$

The normal ordered operators are

$$:H: = \int \frac{d^3\bar{p}}{(2\pi)^3} \sum_s \omega_{\bar{p}} \left(a^{s\dagger}_{\bar{p}} a^s_{\bar{p}} + b^{s\dagger}_{\bar{p}} b^s_{\bar{p}} \right)$$ ← Hamiltonian

$$:\bar{P}: = \int d^3\bar{x}\, \psi^\dagger (-i\bar{\nabla}) \psi$$ ← total momentum

$$= \int \frac{d^3\bar{p}}{(2\pi)^3} \sum_s \bar{p} \left(a^{s\dagger}_{\bar{p}} a^s_{\bar{p}} + b^{s\dagger}_{\bar{p}} b_{\bar{p}} \right)$$

$$:Q: = \int \frac{d^3\bar{p}}{(2\pi)^3} \sum_s \left(a^{s\dagger}_{\bar{p}} a^s_{\bar{p}} - b^{s\dagger}_{\bar{p}} b_{\bar{p}} \right)$$ ← total charge

You will study this more in the problem set

Lorentz invariance of two-point functions (again)

Schwartz 12.4.2

We saw that, for complex scalar fields, we had to impose commutation relations for the time-ordered two-point function to be manifestly Lorentz invariant.

Let's see how this works for spinors.

First we start with

$$\langle 0|\psi(0)\bar{\psi}(x)|0\rangle = \int \frac{d^3\bar{p}}{(2\pi)^3}\int \frac{d^3\bar{q}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\bar{p}}}\sqrt{2\omega_{\bar{q}}}}$$

$$\times \sum_{s,s'} \langle 0|\left(a^s_{\bar{p}}u^s_{\bar{p}} + b^{s\dagger}_{\bar{p}}v^s_{\bar{p}}\right)\left(a^{s'\dagger}_{\bar{q}}\bar{u}^{s'}_{\bar{q}}e^{iq\cdot x} + b^{s'}_{\bar{q}}\bar{v}^{s'}_{\bar{q}}e^{-iq\cdot x}\right)|0\rangle$$

$$= \int \frac{d^3\bar{p}}{(2\pi)^3}\int\frac{d^3\bar{q}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\bar{p}}}}\frac{1}{\sqrt{2\omega_{\bar{q}}}}\sum_{s,s'} u^s_{\bar{p}}\bar{u}^{s'}_{\bar{q}}\,\underbrace{\langle 0|a^s_{\bar{p}}a^{s'\dagger}_{\bar{q}}|0\rangle}_{=(2\pi)^3\delta^{(3)}(\bar{p}-\bar{q})\delta^{ss'}} e^{iq\cdot x}$$

$$= \int\frac{d^3\bar{p}}{(2\pi)^3}\frac{1}{2\omega_{\bar{p}}}\sum_s u^s_{\bar{p}}\bar{u}^s_{\bar{p}}e^{ip\cdot x}$$

now recall our "outer products"

$$\sum_{s=1}^{2} u_s(p)\bar{u}_s(p) = \not{p} + m$$

$$= \int\frac{d^3\bar{p}}{(2\pi)^3}\frac{1}{2\omega_{\bar{p}}}e^{ip\cdot x}(\not{p}+m)$$

$$= \underbrace{(-i\not{\partial}+m)}_{-(i\not{\partial}-m)}\int\frac{d^3\bar{p}}{(2\pi)^3}\frac{e^{ip\cdot x}}{2\omega_{\bar{p}}}$$

We also have

$$\langle 0|\bar{\varphi}(x)\,\varphi(0)|0\rangle = \sum_{ss'}\int\frac{d^3\bar{p}}{(2\pi)^3}\int\frac{d^3\bar{q}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\bar{p}}}}\frac{1}{\sqrt{2\omega_{\bar{q}}}}$$

$$\times\langle 0|\left(b^{s'}_{\bar{q}}\bar{v}^{s'}_{\bar{q}}e^{-iq\cdot x}+a^{s'\dagger}_{\bar{q}}\bar{u}^{s'}_{\bar{q}}e^{iq\cdot x}\right)\left(a^{s}_{\bar{p}}u^{s}_{\bar{p}}+b^{s\dagger}_{\bar{p}}v^{s}_{\bar{p}}\right)|0\rangle$$

$$= \int\frac{d^3\bar{p}}{(2\pi)^3}\frac{1}{2\omega_{\bar{p}}}e^{-ip\cdot x}(\not{p}-m) \qquad \sum_{s=1}^{2}v_s(p)\bar{v}_s(p)=\not{p}-m$$

$$= \underbrace{(i\not{\partial}-m)}_{=-(-i\not{\partial}+m)}\int\frac{d^3\bar{p}}{(2\pi)^3}\frac{e^{-ip\cdot x}}{2\omega_{\bar{p}}}$$

What happens if we now try to impose commutation relations?

We have

$$\langle 0|T\{\varphi(0)\bar{\varphi}(x)\}|0\rangle = (i\not{\partial}-m)\int\frac{d^3\bar{p}}{(2\pi)^3}\frac{1}{2\omega_p}\left[e^{-ip\cdot x}\theta(x^0)-e^{ip\cdot x}\theta(-x^0)\right]$$

$$= \varphi(0)\bar{\varphi}(x)\theta(x^0)+\bar{\varphi}(x)\varphi(0)\theta(-x^0)$$

$$= (-i\not{\partial}+m)\int\frac{d^4p}{(2\pi)^4}\frac{p^0}{\sqrt{\bar{p}^2+m^2}}\frac{i}{p^2-m^2+i\epsilon}e^{ip\cdot x}$$

Not Lorentz invariant.

But when we impose anticommutation relations (as we know we should) then we obtain

$$\langle 0|T\{\varphi(0)\bar{\varphi}(x)\}|0\rangle = (-i\not{\partial}+m)\int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{ip\cdot x}$$

$$= \varphi(0)\bar{\varphi}(x)\theta(x^0)-\bar{\varphi}(x)\varphi(0)\theta(-x^0)$$

Lorentz invariant!

In fact, this allows us to write the propagator for our free fermion

$$\boxed{\langle 0|T\{\psi(0)\bar{\psi}(x)\}|0\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon} e^{ip\cdot x}}$$

Remember - the "$i\epsilon$ prescription" fixes the choice of contour and corresponds to the time-ordered product of fields (the most useful two-point function).

To summarise:

- for a scalar $\langle 0|\phi^*(x)\,\phi(0)|0\rangle = \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{e^{-ip\cdot x}}{2\omega_{\bar{p}}}$

$$\langle 0|\phi(0)\,\phi^*(x)|0\rangle = \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{e^{ip\cdot x}}{2\omega_{\bar{p}}}$$

- for a fermion $\langle 0|\bar{\psi}(x)\,\psi(0)|0\rangle = \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{\not{p}}{2\omega_{\bar{p}}} e^{-ip\cdot x}$

$$\langle 0|\psi(0)\,\bar{\psi}(x)|0\rangle = \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{\not{p}}{2\omega_{\bar{p}}} e^{+ip\cdot x}$$

We see that under a rotation $\not{p} \to -\not{p}$, so when we combine these in a time-ordered product there is an extra minus sign. Sending $\not{p} \to -\not{p}$ is a rotation by π, which, for a spinor, corresponds to multiplying by $e^{i\pi/2} = i$. Since we have two fermions we have $e^{i\pi/2}\cdot e^{i\pi/2} = -1$. This is directly related to the fact that spinors embed spin $1/2$ particles!