Spin and statistics

Identical particles

Schwartz 12.1

Remember that particles transform under unitary irreducible representations of the Poincaré group. This means that identical particles must transform in the same representation and have identical additional quantum numbers.

We can create multi-particle states through

$$|\cdots s_1 \bar{p}_1 \overset{\text{no index}}{n} \cdots s_2 \bar{p}_2 n \cdots \rangle = \cdots \sqrt{2\omega_1}\, a^\dagger_{\bar{p}_1 s_1 n} \cdots \sqrt{2\omega_2}\, a^\dagger_{\bar{p}_2 s_2 n} \cdots |0\rangle$$

with normalisation

$$\langle s_i \bar{p}_i n_i \ldots | s_i' \bar{p}_i' n_i' \ldots \rangle = \prod_i \delta_{n_i n_i'} \delta_{s_i s_i'} \; 2\omega_i (2\pi)^3 \delta^{(3)}(\bar{p}_i - \bar{p}_i')$$

If we had acted with these operators in a different order:

$$|\cdots s_2 \bar{p}_2 n \; \cdots s_1 \bar{p}_1 n \cdots \rangle = \cdots \sqrt{2\omega_2}\, a^\dagger_{\bar{p}_2 s_2 n} \cdots \sqrt{2\omega_1}\, a^\dagger_{\bar{p}_1 s_1 n} \cdots |0\rangle$$

Must be the same physical state as above!
$\Rightarrow$ can only differ by a normalisation
But we fixed the normalisation
$\Rightarrow$ can only differ by a phase $\underset{\mathbb{C}}{\alpha} = e^{i\overset{\mathbb{R}}{\vartheta}}$

The phase can only depend on the particle species (n) and not on $\bar{p}$ or $\bar{s}$ (there are no nontrivial one-dimensional representations of the proper Lorentz group), nor on the path by which particles are interchanged (there are no Lorentz invariants to characterise this).

Now we swap the particles back, to give

$$|\cdots s_1 \bar{p}_1 n \cdots s_2 \bar{p}_2 n \cdots\rangle = \alpha_n^2 |\cdots s_1 \bar{p}_1 n \cdots s_2 \bar{p}_2 n \cdots\rangle$$

Clearly $\alpha_n = \pm 1$

- $\alpha_n = +1$ — bosons
  satisfy Bose-Einstein statistics
- $\alpha_n = -1$ — fermions
  satisfy Fermi-Dirac statistics

every particle is either a boson or a fermion!

For bosons

$$a^+_{\bar{p}_1 s_1 n} a^+_{\bar{p}_2 s_2 n} |\psi\rangle = a^+_{\bar{p}_2 s_2 n} a^+_{\bar{p}_1 s_1 n} |\psi\rangle \qquad \forall |\psi\rangle$$

$$\Rightarrow \left[a^+_{\bar{p}_1 s_1 n}, a^+_{\bar{p}_2 s_2 n}\right] = \left[a_{\bar{p}_1 s_1 n}, a_{\bar{p}_2 s_2 n}\right] = 0$$

Then, using $\langle \bar{p}_1 | \bar{p}_2 \rangle = 2\omega_1 (2\pi)^3 \delta^{(3)}(\bar{p}_1 - \bar{p}_2)$ we can also show that

$$[a_{\bar{p}_1 s_1 n}, a^{+}_{\bar{p}_2 s_2 n}] = (2\pi)^3 \delta^{(3)}(\bar{p}_1 - \bar{p}_2)\, \delta_{s_1 s_2}$$

recognise this ?!

For fermions

$$\{a^{+}_{\bar{p}_1 s_1 n}, a^{+}_{\bar{p}_2 s_2 n}\} = \{a_{\bar{p}_1 s_1 n}, a_{\bar{p}_2 s_2 n}\} = 0$$

$$\{a^{+}_{\bar{p}_1 s_1 n}, a_{\bar{p}_2 s_2 n}\} = (2\pi)^3 \delta^{(3)}(\bar{p}_1 - \bar{p}_2)\, \delta_{s_1 s_2}$$

this leads directly to the Pauli exclusion principle

$$a^{+}_{\bar{p}} a^{+}_{\bar{p}} |0\rangle = - a^{+}_{\bar{p}} a^{+}_{\bar{p}} |0\rangle = 0$$

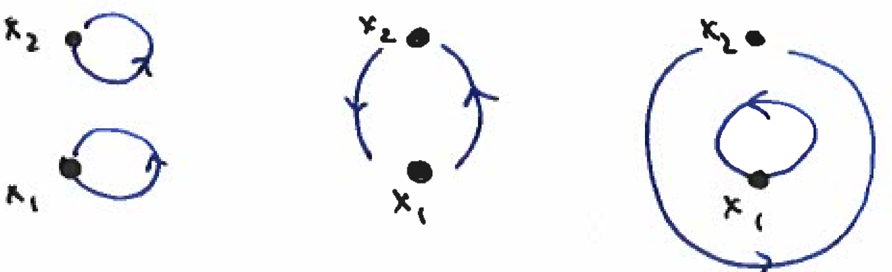
The existence of identical particles (an observational fact) is a direct consequence of writing fields in terms of creation and annihilation operators. One of the most useful motivations for us to use creation and annihilation operators is that this ensures that two asymptotically distant measurements cannot affect each other.

the "cluster decomposition principle"

Path dependence

$\boxed{\text{Schwartz 12.2}}$

Let's now think about the actual, physical process of swapping particles. We can do this in many ways, but we can always characterise the process in terms of the angle one particle rotates around another (no matter the path taken in between observations)



no exchange $\vartheta = 0$ | exchange $\vartheta = \pi$ | no exchange $\vartheta = 2\pi$

← note: we can't tell any of these situations apart! Because the particles are <u>identical</u>.

In principle, the particles could pick up a phase that depends on this angle

$$S|\phi_1(x_1)\,\phi_2(x_2)\rangle = e^{i\vartheta K}|\phi_2(x_1)\,\phi_1(x_2)\rangle$$

↑ "switching" operator

In three spatial dimensions, we expect the same results for ϑ and $\vartheta = 2\pi$ $\Rightarrow$ K is an integer $(K \in \mathbb{Z})$

Then, for $\vartheta = \pi$, we can <u>only</u> have

$$S|\phi_1(x_1)\,\phi_2(x_2)\rangle = \pm|\phi_2(x_1)\,\phi_1(x_2)\rangle$$

← <u>only</u> fermion or boson statistics allowed!

Thus, only the first two paths are possible in our example figure. Either the particles are swapped, or they are not swapped.

The second path can be obtained through: ← nothing happens to state in a free theory

← ie act with Poincaré transformation!

1. translate particles by distance $|\bar{x}_2 - \bar{x}_1|$
2. rotate system by π to move ϕ_2 back to $\bar{x}_1$.

for scalars (no spin) – rotation does nothing

for spinors – rotation is $\Lambda_S(\pi) = \begin{pmatrix} i & & & \\ & -i & & \\ & & i & \\ & & & -i \end{pmatrix}$

$$\Rightarrow \varphi_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} i \\ 0 \\ i \\ 0 \end{pmatrix} = i\varphi_1$$

The effects are:

- scalars $\quad S|\phi_1(x_1)\, \phi_2(x_2)\rangle = +\phi_2(x_1)\, \phi_1(x_2)\rangle$
- spinors $\quad S|\varphi_1(x_1)\, \varphi_2(x_2)\rangle = -|\varphi_2(x_1)\, \varphi_1(x_1)\rangle$

Note:

- this derivation works for any half-integer or integer spin
- only requirement: half-integer spin particles go to minus themselves under 2π rotation
- arises because $SL(2,\mathbb{C})$, the universal cover of the Lorentz group, is simply connected, while the Lorentz group is doubly connected

Lorentz invariance of two-point functions

Schwartz 12.4

By studying time ordered two-point functions, we will see that Lorentz invariance *requires* a close connection between spin and statistics.

In fact the arguments below apply more generally to the S-matrix, which we haven't met yet.

Spin 0

For a complex scalar, we have

$$\langle 0|T\{\phi^*(x)\,\phi(0)\}|0\rangle = \langle 0|\phi^*(x)\,\phi(0)|0\rangle\,\theta(x^0) + \langle 0|\phi(0)\,\phi^*(x)|0\rangle\,\theta(-x^0)$$

1st term is

$$\langle 0|\phi^*(x)\phi(0)|0\rangle = \int\frac{d^3\bar{p}}{(2\pi)^3}\int\frac{d^3\bar{q}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\bar{p}}}}\cdot\frac{1}{\sqrt{2\omega_{\bar{q}}}}\langle 0|\left(a_{\bar{p}}^{\dagger}e^{ip\cdot x} + b_{\bar{p}}e^{-ip\cdot x}\right) \times \left(a_{\bar{q}} + b_{\bar{q}}^{\dagger}\right)|0\rangle$$

$$= \int\frac{d^3\bar{p}}{(2\pi)^3}\int\frac{d^3\bar{q}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\frac{1}{\sqrt{2\omega_q}}\langle\bar{p}|\frac{e^{-ip\cdot x}}{\sqrt{2\omega_p}}\cdot\frac{1}{\sqrt{2\omega_q}}|\bar{q}\rangle$$

$$= \int\frac{d^3\bar{p}}{(2\pi)^3}\int\frac{d^3\bar{q}}{(2\pi)^3}\frac{e^{-ip\cdot x}}{4\omega_p\omega_q}\,2\omega_{\bar{p}}\,(2\pi)^3\,\delta^{(3)}(\bar{p}-\bar{q})$$

$$= \int\frac{d^3\bar{p}}{(2\pi)^3}\frac{1}{2\omega_{\bar{p}}}\,e^{-ip\cdot x}$$

2nd term is

$$\langle 0|\phi(0)\,\phi^*(x)|0\rangle = \int\frac{d^3\bar{p}}{(2\pi)^3}\frac{1}{2\omega_{\bar{p}}}\,e^{ip\cdot x}$$

This means we have

$$\langle 0|T\{\phi^*(x)\,\phi(0)\}|0\rangle = \int \frac{d^3\bar{p}}{(2\pi)^3}\frac{1}{2\omega_p}\Big[e^{-i\omega_p x^0 + i\bar{p}\cdot\bar{x}}\,\vartheta(x^0) + e^{i\omega_p x^0 - i\bar{p}\cdot\bar{x}}\,\vartheta(-x^0)\Big]$$

$$= \int \frac{d^3\bar{p}}{(2\pi)^3}\frac{e^{-i\bar{p}\cdot\bar{x}}}{2\omega_{\bar{p}}}\Big[e^{-i\omega_p x^0}\vartheta(x^0) + e^{i\omega_p x^0}\,\vartheta(-x^0)\Big]$$

(*)

$$= \int \frac{d^4p}{(2\pi)^4}\frac{i}{2\omega_{\bar{p}}}\,e^{ip\cdot x}\left(\frac{1}{p^0-(\omega_p - i\epsilon)} - \frac{1}{p^0-(-\omega_{\bar{p}}+i\epsilon)}\right)$$

$$= i\int \frac{d^4p}{(2\pi)^4}\frac{e^{ip\cdot x}}{p^2-m^2+i\epsilon}$$

(*) - take $\bar{p} \to -\bar{p}$ in first term

<u>manifestly</u> Lorentz invariant!

Aside: Schwartz's proof of (*) requires

$$e^{i\omega_{\bar{p}}t}\,\vartheta(-t) = \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{\omega-(\omega_{\bar{p}}-i\epsilon)}\,e^{i\omega t}$$

$$e^{-i\omega_{\bar{p}}t}\,\vartheta(t) = -\frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{\omega-(-\omega_{\bar{p}}+i\epsilon)}\,e^{i\omega t}$$

This follows from

$$e^{-i\omega_{\bar{p}}t}\,\vartheta(t) + e^{i\omega_{\bar{p}}t}\,\vartheta(-t) = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\left(\frac{-2\omega_{\bar{p}}}{2\pi i}\right)\frac{d\omega}{\omega^2-\omega_{\bar{p}}^2+i\epsilon}\,e^{i\omega t}$$

which we prove by writing the integrand as

$$\frac{e^{i\omega t}}{\omega^2-\omega_p^2+i\epsilon} = \frac{e^{i\omega t}}{(\omega-(\omega_{\bar{p}}-i\epsilon))(\omega-(-\omega_{\bar{p}}+i\epsilon))} = \frac{e^{i\omega t}}{2\omega_{\bar{p}}}\left(\frac{1}{\omega-(\omega_{\bar{p}}-i\epsilon)} - \frac{1}{\omega-(-\omega_{\bar{p}}-i\epsilon)}\right)$$

$+\,O(\epsilon^2)$

We close the contour upwards for $t > 0$ and downwards for $t < 0$. Then the first term is nonzero only for $t<0$

$$\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega - (\omega_{\bar{p}} - i\epsilon)} d\omega = \underset{\substack{\uparrow \\ \text{clockwise contour !}}}{-2\pi i}\, e^{i\omega_{\bar{p}} t}\, \theta(-t) + O(\epsilon)$$

The second term is nonzero only for $t > 0$

$$\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega + (\omega_p - i\epsilon)} d\omega = 2\pi i \, e^{-i\omega_p t}\, \theta(t) + O(\epsilon)$$

Suppose instead we took our complex scalars to satisfy <u>anticommutation</u> relations (note: they do not!). For consistency with anticommutation relations, we must then define time ordering as Q: why ?!

$$T\{\phi^*(x)\,\phi(0)\} = \phi^*(x)\,\phi(0)\,\theta(x^0) - \phi(0)\,\phi^*(x)\,\theta(-x^0)$$

Then we would have

$$\langle 0|T\{\phi^*(x)\phi(0)\}|0\rangle = \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{2\omega_{\bar{p}}} \left[e^{-ip\cdot x}\,\theta(x^0) - e^{ip\cdot x}\,\theta(-x^0)\right]$$

$\downarrow$ let $x^0 = t$

$$= \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{e^{-i\bar{p}\cdot\bar{x}}}{2\omega_{\bar{p}}} \left[e^{-i\omega_{\bar{p}} t}\,\theta(t) - e^{i\omega_{\bar{p}} t}\,\theta(-t)\right]$$

$$= \int \frac{d^4 p}{(2\pi)^4} \frac{e^{ip\cdot x}}{2\omega_{\bar{p}}} \left[\frac{-i}{\omega - (-\omega_p + i\epsilon)} - \frac{i}{\omega - (\omega_{\bar{p}} - i\epsilon)}\right]$$

$$= \int \frac{d^4 p}{(2\pi)^4} \frac{p^0}{\sqrt{\bar{p}^2 + m^2}} \frac{-i}{p^2 - m^2 + i\epsilon}\, e^{ip\cdot x}$$

$\nwarrow$ manifestly not Lorentz invariant !