Causality

P+S 2.4
Schwartz 6.2

free! = non-interacting

see also
Schwartz 12.6

We've constructed a theory of quantum fields that have a relativistic dispersion relation (the Klein-Gordon equation).

But we haven't formulated everything in a Lorentz covariant way

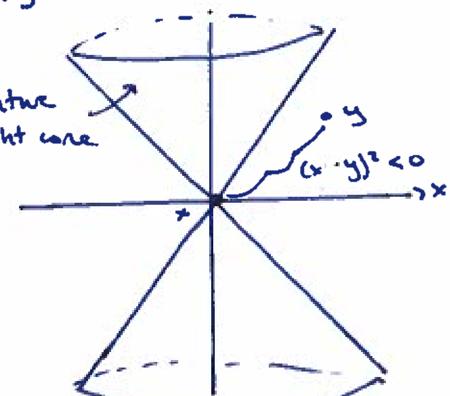
recall equal-time commutation relations $[\hat{\phi}(\bar{x},t), \hat{\pi}(\bar{y},t)] = i\delta^{(3)}(\bar{x}-\bar{y})$

We would like to formulate things for arbitrary times.

But our theory must respect causality

$\Rightarrow$ in QFT this means $[\hat{O}_1(x), \hat{O}_2(y)] = 0 \quad \forall\, (x-y)^2 < 0$

If this is not true then a measurement at x could affect a measurement at y, even though they are not causally connected.



Simplest example we can consider

$$[\hat{\phi}(x), \hat{\phi}(y)] = \int \frac{d^3k\, d^3p}{(2\pi)^6} \frac{1}{2\sqrt{E_k E_p}} \Big(e^{-ikx+ipy} [\hat{a}(\bar{k}), \hat{a}^\dagger(\bar{p})] + e^{ik\cdot x - ip\cdot y} [\hat{a}^\dagger(\bar{k}), \hat{a}(\bar{p})] \Big)$$

$\leftarrow (2\pi)^3 \delta^{(3)}(\bar{k}-\bar{p})$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} \Big(e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)} \Big)$$

$$\equiv \Delta(x-y) - \Delta(y-x)$$

Here

$$\Delta(z) \equiv \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} e^{-ik\cdot z}$$

which we know is Lorentz covariant

- this object is a $\mathbb{C}$-number — true only for free theories
- non-zero commutator for $(x-y)^2 \geq 0$
- vanishing commutator for $(x-y)^2 < 0$

} ✓

To see these last two properties

1. take $\bar{x}=0,\ x^0=0$ and $\bar{y}=0,\ y^0-x^0 \equiv t>0$

always possible to find Lorentz transformation to these coordinates from $y \in$ future lightcone

$$\Delta(0,t) = \int \frac{d^3\bar{k}}{(2\pi)^3}\frac{1}{2\sqrt{|\bar{k}|^2+m^2}}e^{-it\sqrt{|\bar{k}|^2+m^2}}$$

$$= \frac{4\pi}{(2\pi)^3}\int_0^\infty dk \frac{k^2}{2\sqrt{k^2+m^2}}e^{-it\sqrt{k^2+m^2}}$$

$$= \frac{1}{4\pi}\int_m^\infty dE\sqrt{E^2-m^2}\,e^{-iEt}$$

$$\underset{t\to\infty}{\sim} e^{-imt}$$

Then $[\phi(x),\phi(y)] \sim e^{-imt}-e^{imt} \neq 0 \qquad (x-y)^2 \geq 0$

2. take $\bar{x}=0,\ x^0=0$ and $\bar{x}-\bar{y}=-\bar{r},\ y^0=0$

again always possible if $(x-y)^2<0$

$$\Delta(\bar{r},0) = \int\frac{d^3k}{(2\pi)^3}\frac{1}{2\sqrt{|\bar{k}|^2+m^2}}e^{+i\bar{k}\cdot\bar{r}}$$

$$= \frac{1}{2}\frac{1}{(2\pi)^3}\int_0^\infty dk\,k^2\frac{1}{\sqrt{k^2+m^2}}\int_{-1}^{1}d(\cos\theta)\int_0^{2\pi}d\phi\,e^{ikr\cos\theta} \qquad \boxed{r=|\bar{r}|}$$

$$= \frac{1}{2(2\pi)^3}\int_0^\infty dk\frac{k^2}{\sqrt{k^2+m^2}}\frac{2\pi}{ikr}\left(e^{ikr}-e^{-ikr}\right)$$

$$= \frac{-i}{8\pi^2 r}\int_0^\infty dk\frac{k}{\sqrt{k^2+m^2}}\left(e^{ikr}-e^{-ikr}\right) \qquad \} \ k\to k'=-k$$

$$= \frac{-i}{8\pi^2 r}\left(\int_0^\infty dk\frac{k e^{ikr}}{\sqrt{k^2+m^2}} - \int_0^{-\infty}dk\frac{k e^{ikr}}{\sqrt{k^2+m^2}}\right)$$

$$= \frac{-i}{8\pi^2 r}\int_{-\infty}^{\infty}dk\frac{k e^{ikr}}{\sqrt{k^2+m^2}}$$

$$\underset{r\to\infty}{\sim} e^{-mr} \neq 0$$ see P+S pg 27-28

<u>But</u> $[\phi(x),\phi(y)] = \Delta(\bar{r})-\Delta(-\bar{r}) \sim (e^{-mr}-e^{-mr}) = 0 \qquad (x-y)^2<0$

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In general, the way that we probe a QFT (ie. make "measurements") in order to understand what is going on is by calculating <u>correlation functions</u>, which tell us about correlations (surprise surprise!) between different parts of our system. In other words, how does one part of our system affect another?

The simplest such correlation function we can consider (that isn't zero) is the <u>two-point function</u>

$$\langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle = \int\frac{d^3\bar{p}\,d^3\bar{q}}{(2\pi)^6}\frac{1}{\sqrt{4E_pE_q}}\langle 0|\hat{a}(\bar{p})\hat{a}^+(\bar{q})|0\rangle e^{-ip\cdot x+iq\cdot y}$$

only nonvanishing term in expansion in terms of $\hat{a},\hat{a}^+$

Now

3-vectors

$$\langle 0|\hat{a}(\bar{p})\hat{a}^+(\bar{q})|0\rangle = \langle\bar{p}|\bar{q}\rangle = (2\pi)^3\delta^{(3)}(\bar{p}-\bar{q})$$

N.B. $\langle p|q\rangle = 2E_p(2\pi)^3\delta^{(3)}(\bar{p}-\bar{q})$

4-vectors!

recall our discussion of the normalisation of states

so

$$\langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip(x-y)}$$

$$= \Delta(x-y)$$

we saw this before!

This object is called a <u>propagator</u>

But as we saw it is not as useful as the combination $[\hat{\phi}(x),\hat{\phi}(y)]$ because it does not vanish outside the light cone.

Propagator ~ probability amplitude for a particle to travel from x to y

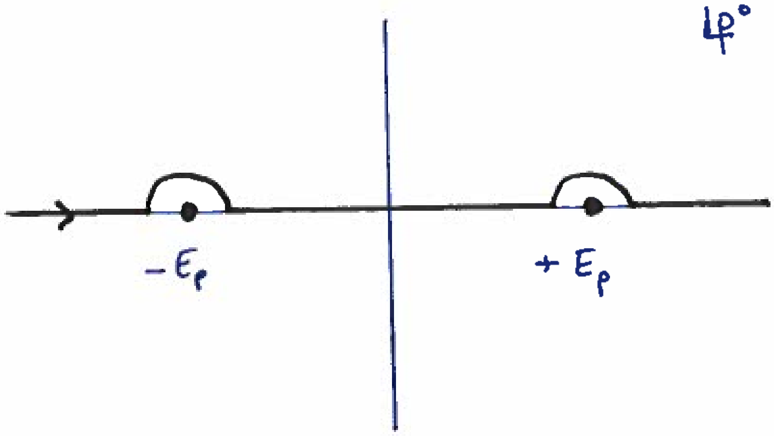
Let's return to the combination that <u>does</u> satisfy causality

$$\langle 0|[\phi(x),\phi(y)]|0\rangle = \int\frac{d^3\bar{p}}{(2\pi)^3}\frac{1}{2E_p}\left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right)$$

$$= \int\frac{d^3\bar{p}}{(2\pi)^3}\left\{\frac{1}{2E_p}e^{-ip\cdot(x-y)}\Big|_{p^0=E_p} + \frac{1}{2(-E_p)}e^{-ip\cdot(x-y)}\Big|_{p^0=-E_p}\right\}$$

$$\underset{x^0>y^0}{=} \int\frac{d^3\bar{p}}{(2\pi)^3}\int\frac{dp^0}{2\pi i}\frac{-1}{p^2-M^2}e^{-ip\cdot(x-y)}$$

In the last line, we use the contour



$x^0 > y^0$: close in lower half plane
$\Rightarrow$ non zero

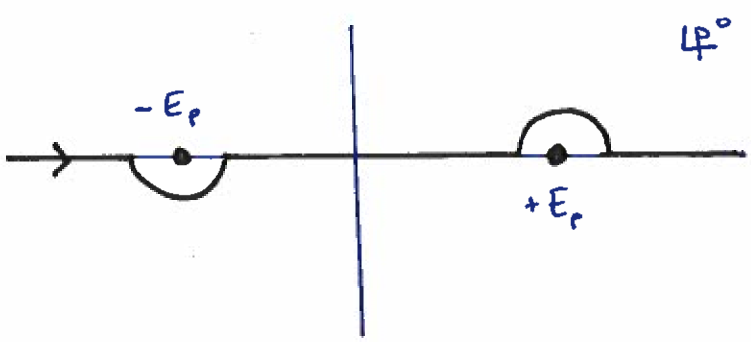
$x^0 < y^0$: close in upper half plane
$\Rightarrow$ zero

This object is another kind of propagator

$$D_{KG}(x-y) \equiv \theta(x^0-y^0)\langle 0|[\hat{\phi}(x),\hat{\phi}(y)]|0\rangle$$

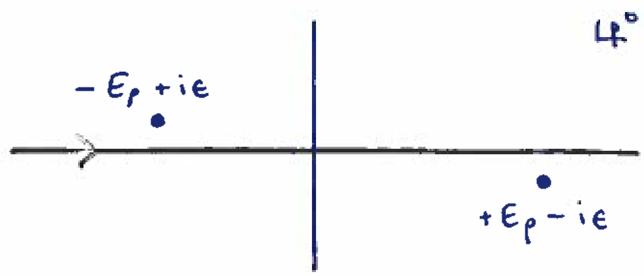
$$\theta(x^0-y^0) = \begin{cases} 0 & x^0<y^0 \\ 1 & x^0>y^0 \end{cases}$$

But this is not the only choice of contour – there are four and each choice specifies a different function. The most useful will prove to be the <u>Feynman propagator</u>, defined by



which is entirely equivalent to

ie. move the poles not the contour!



Let's study this in more detail

The Feynman propagator corresponds to the correlation function of the time-ordered product of fields

different from normal ordering!

Time ordering: place all fields in time-order, with the earliest to the right

$$T\{\hat{\phi}(\bar{x},t_1)\,\hat{\phi}(\bar{y},t_2)\} \equiv \vartheta(t_1-t_2)\,\hat{\phi}(\bar{x},t_1)\,\hat{\phi}(\bar{y},t_2) + \vartheta(t_2-t_1)\,\hat{\phi}(\bar{y},t_2)\,\hat{\phi}(\bar{x},t_1)$$

In fact

$$\begin{aligned} D_F(x-y) &= \langle 0|T\{\phi(x)\,\phi(y)\}|0\rangle \\ &= \theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0-x^0)\langle 0|\phi(y)\phi(x)|0\rangle \\ &= \lim_{\epsilon\to 0^+}\int \frac{d^4p}{(2\pi)^4}\,\frac{i}{(p^2-m^2+i\epsilon)}\,e^{-ip\cdot(x-y)} \end{aligned}$$

To show the last step, let's consider the pole structure

$$|\epsilon \ll 1|$$

$$\begin{aligned} p^2-m^2+i\epsilon &= (p^0)^2-\bar{p}^2-m^2+i\epsilon \\ &= (p^0)^2-E_p^2+i\epsilon \\ &= \left(p^0-\sqrt{E_p^2-i\epsilon}\right)\left(p^0+\sqrt{E_p^2-i\epsilon}\right) \\ &\simeq \left(\underbrace{p^0-(E_p-i\epsilon)}+O(\epsilon^2)\right)\left(p^0+\underbrace{(E_p-i\epsilon)}+O(\epsilon^2)\right) \end{aligned}$$

poles as in diagram on previous page

Now let's take $x^0 > y^0$

$$\Rightarrow e^{-ip^0(x^0-y^0)} = e^{-i\,\mathrm{Re}(p^0)\,|x^0-y^0|}\,e^{+\mathrm{Im}(p^0)\,|x^0-y^0|}$$

$\Rightarrow$ we close the contour in the lower half plane (when $\mathrm{Im}(p^0)<0$) and the contour at infinity contributes zero

So our integral becomes

$$\int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon} = i\int \frac{d^3\bar{p}}{(2\pi)^3} e^{+i\bar{p}\cdot(\bar{x}-\bar{y})} \int_{-\infty}^{\infty} \frac{dp^0}{2\pi} \frac{e^{-ip^0(x^0-y^0)}}{(p^0 - (E_p - i\epsilon))(p^0 + (E_p - i\epsilon))}$$

$$= i\int \frac{d^3\bar{p}}{(2\pi)^3} e^{+i\bar{p}\cdot(\bar{x}-\bar{y})} (-2\pi i) \underset{p^0 = E_p - i\epsilon}{\text{Residue}} \left\{ \int \frac{dp^0}{2\pi} \frac{e^{-ip^0(x^0-y^0)}}{(p^0 - (E_p - i\epsilon))(p^0 + (E_p - i\epsilon))} \right.$$

$$= i\int \frac{d^3\bar{p}}{(2\pi)^3} e^{+i\bar{p}\cdot(\bar{x}-\bar{y})} \left(-\frac{2\pi i}{2\pi}\right) \frac{e^{-i(E_p - i\epsilon)(x^0-y^0)}}{2(E_p - i\epsilon)}$$

clockwise contour $\Rightarrow -2\pi i$!

$$\underset{\epsilon\to 0}{=} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x-y)} = \Delta(x-y)$$

Next we take $y^0 > x^0$

$$\Rightarrow e^{-ip^0(x^0-y^0)} = e^{ip^0|y^0-x^0|} = e^{i\,\mathrm{Re}(p^0)|y^0-x^0|} e^{-\mathrm{Im}(p^0)|y^0-x^0|}$$

$\Rightarrow$ we close the contour in the upper half plane (where $\mathrm{Im}(p^0) > 0$) and again the contour at infinity gives zero

In this case our integral becomes

$$\int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon} = i\int \frac{d^3\bar{p}}{(2\pi)^3} e^{+i\bar{p}\cdot(\bar{x}-\bar{y})} \int_{-\infty}^{\infty} \frac{dp^0}{2\pi} \frac{e^{-ip^0(x^0-y^0)}}{(p^0 - (E_p - i\epsilon))(p^0 + (E_p - i\epsilon))}$$

$$= i\int \frac{d^3\bar{p}}{(2\pi)^3} e^{+i\bar{p}\cdot(\bar{x}-\bar{y})} (2\pi i) \underset{p^0 = -E_p + i\epsilon}{\text{Residue}} \left\{ \int_{-\infty}^{\infty} \frac{dp^0}{2\pi} \frac{e^{-ip^0(x^0-y^0)}}{(p^0 - (E_p - i\epsilon))(p^0 + (E_p - i\epsilon))} \right\}$$

$$= i\int \frac{d^3\bar{p}}{(2\pi)^3} e^{+i\bar{p}\cdot(\bar{x}-\bar{y})} \frac{2\pi i}{2\pi} \frac{e^{-i(-E_p + i\epsilon)(x^0-y^0)}}{-2(E_p - i\epsilon)}$$

$$= \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{2E_p} e^{+i\bar{p}\cdot(\bar{x}-\bar{y})} \underbrace{e^{ip^0(x^0-y^0)}}_{e^{ip^0(x^0-y^0)} = e^{-ip^0(y^0-x^0)}} = \Delta(y-x)$$

Therefore

$$\lim_{\epsilon\to 0^+} \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)} = \theta(x^0 - y^0)\Delta(x-y) + \theta(y^0 - x^0)\Delta(y-x)$$

$$= D_F(x-y)$$

#

There is another interpretation of the Feynman propagator that illustrates its utility. Note:

$$(\partial_x^2 + m^2) D_F(x-y) = (\partial_x^2 + m^2)\, i \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon}$$

$$= i \int \frac{d^4p}{(2\pi)^4} \frac{(-p^2+m^2)}{p^2-m^2+i\epsilon} e^{-ip\cdot(x-y)}$$

$$= -i \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)}$$

$$= -i\, \delta^{(4)}(x-y)$$

$\epsilon \to 0$ limit will be always understood in future

$\Rightarrow$ Feynman propagator is a Green's function of the Klein-Gordon operator

See P+S pg. 30 for proof in time-ordered form (careful with the θ-function!).

So are D_{KG} and Δ, but D_F is the most useful

A summary

- Propagators are Green's functions of the Klein-Gordon operator.

$$O_{KG} \cdot D(x-y) \propto \delta^{(4)}(x-y)$$

- Two-point function

$$\langle 0 | \hat{\phi}(x) \hat{\phi}(y) | 0 \rangle = \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x-y)} \equiv \Delta(x-y)$$

- Klein-Gordon propagator

$$\langle 0 | [\hat{\phi}(x), \hat{\phi}(y)] | 0 \rangle = \Delta(x-y) - \Delta(y-x)$$

$$D_{KG}(x-y) \equiv \theta(x^0 - y^0) \langle 0 | [\hat{\phi}(x), \hat{\phi}(y)] | 0 \rangle$$

$$\underset{x^0 - y^0 > 0}{=} \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip\cdot(x-y)}$$

- Feynman propagator

$$\langle 0 | T\{ \hat{\phi}(x), \hat{\phi}(y) \} | 0 \rangle = \theta(x-y)\Delta(x-y) - \theta(y-x)\Delta(x-y)$$

$$D_F(x-y) = \lim_{\epsilon \to 0^+} \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)}$$