The Vacuum and normal ordering

Tong 2.3

There are actually two sources of divergence here

IR = long wavelength = large scale

- infrared (IR)

The Hamiltonian will give an infinite vacuum energy:

$$H|0\rangle = \underbrace{\int \frac{d^3\bar{p}}{(2\pi)^3} \frac{\omega_{\bar{p}}}{2} (2\pi)^3 \delta^{(3)}(\bar{0})}_{\text{Infinite "}E_0\text{"}} |0\rangle$$

Q: what happened to the other term in H?

and the $\delta^{(3)}(\bar{0})$ part of this comes from the fact we've assumed the Universe is infinitely large. We can deal with this by using a finite box

$$(2\pi)^3 \delta^{(3)}(\bar{0}) = \lim_{L\to\infty} \int_{-L/2}^{L/2} d^3\bar{x}\, e^{i\bar{p}\cdot\bar{x}} \Big|_{\bar{p}=0} = \lim_{L\to\infty} \int_{-L/2}^{L/2} d^3\bar{x} = V$$

volume of our box

and studying the energy density

$$\mathcal{E}_0 = \frac{E_0}{V} = \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{\omega_{\bar{p}}}{2}$$

but this is still infinite!

UV = short wavelength = short scale

- ultraviolet (UV)

The integral $\int d^3\bar{p} \sqrt{\bar{p}^2 + m^2}$ diverges at $|\bar{p}| \to \infty$.

This UV divergence arises because we have assumed our theory is valid up to some arbitrarily large momentum (small distance scale) - but this is almost certainly not true, so we can remove this divergence by introducing a <u>cutoff</u>

$$\int d^3\bar{p}\,\sqrt{\bar{p}^2+m^2} = \underbrace{\int d\Omega_3}_{=4\pi} \int_0^\infty dp\,\sqrt{p^2+m^2} \;\text{"="}\; \infty \qquad \Lambda_m \equiv \sqrt{\Lambda^2+m^2}$$

$$\longrightarrow 4\pi \int_0^\Lambda dp\,\sqrt{p^2+m^2} = 2\pi\left(\Lambda\cdot\Lambda_m + m^2\log\left(\frac{\Lambda+\Lambda_m}{m}\right)\right)$$

Λ is our cutoff

Note that this result now depends on the cutoff. The method for dealing with this is called renormalisation and is a topic for next semester.

A more "practical" way to tackle these infinities is to introduce <u>normal ordering</u>.

A normal ordered string of operators has all annihilation operators on the right.

$$\hookrightarrow \; :H: = \int \frac{d^3\bar{p}}{(2\pi)^3}\,\omega_{\bar{p}}\, a^\dagger_{\bar{p}}\, a_{\bar{p}}$$

symbol for normal ordering

Generally justified "because we only measure <u>energy differences</u>"

normal ordered operators acting on the vacuum tend to give zero, because the annihilation operators act directly on the vacuum.

Tong discusses a "counterexample", the Casimir effect, but see arXiv:1605.0443 [illegible]

Our Hamiltonian satisfies

$$[H, a_{\bar{p}}^\dagger] = \omega_{\bar{p}} a_{\bar{p}}^\dagger$$
$$[H, a_{\bar{p}}] = -\omega_{\bar{p}} a_{\bar{p}}$$

so when we act on the vacuum, we get exactly what we expect:

- create a state with momentum $\bar{p}$ $\left(|\bar{p}\rangle = \sqrt{2\omega_p}\, a_p^\dagger |0\rangle\right.$
- state has energy

$$H|\bar{p}\rangle = \omega_{\bar{p}} |\bar{p}\rangle \quad \text{with} \quad \omega_p = \sqrt{\bar{p}^2 + m^2}$$

relativistic momentum eigenstate!

We can create multiparticle states by acting multiple times on the vacuum

$$|\bar{p}_1, \dots, \bar{p}_n\rangle \propto a_{\bar{p}_1}^\dagger \dots a_{\bar{p}_n}^\dagger |0\rangle$$

commute, so $|\bar{p}_1, \bar{p}_2\rangle = |\bar{p}_2, \bar{p}_1\rangle$ etc.

$\Rightarrow$ <u>bosons</u>!

The set of all $|p\rangle, |p_1, p_2\rangle, \dots$ is called a Fock space.

We can also introduce the operators

- $N = \int \frac{d^3\bar{p}}{(2\pi)^3} a_{\bar{p}}^\dagger a_{\bar{p}}$ $\quad N|\bar{p}_1, \dots, \bar{p}_n\rangle = n|\bar{p}_1, \dots, \bar{p}_n\rangle$

number operator

- $P^i = -\int d^3x\, \pi \nabla^i \phi = \int \frac{d^3\bar{p}}{(2\pi)^3} p^i a_{\bar{p}}^\dagger a_{\bar{p}}$

total momentum operator

Time dependence

Schwartz 2.3.2

So far, we have been working with equal-time commutation relations and treated time evolution as separate from the spatial dependence. This is not ideal in a relativistic theory. In fact, we have been working in the Schrödinger picture, where the one-particle states evolve with time.

$\leftarrow i\frac{d}{dt}|\bar{p}\rangle_t = H|\bar{p}\rangle_t$

$\Rightarrow |\bar{p}\rangle_t = e^{-i\omega_p t}|\bar{p}\rangle_0$

We want to work with a Lorentz covariant theory, so it is better to work in the Heisenberg picture, in which the operators depend on time as well as space

Recall $\quad O_H(t) = e^{iHt}\, O_S\, e^{-iHt}$ ← Schrödinger picture

with $\quad \frac{d}{dt} O_H(t) = i\left[H, O_H\right]$

Heisenberg picture ... Hamiltonian

We don't usually bother with subscripts, we write $\phi(x)$ or $\phi(\bar{x})$.

In the case of our field operators, we know that the creation/annihilation operators are simple harmonic oscillators so they depend on time as

$$a_p(t) = e^{-i\omega_p t}\, a_p$$

$$a_p^\dagger(t) = e^{i\omega_p t}\, a_p^\dagger$$

Q: why the change of sign with respect to $\phi(\bar{x})$?

$$\Rightarrow \quad \phi(x) = \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x}\right)$$

We can now calculate the equation of motion for our time-dependent free field

$$i\partial_t \phi = [\phi, :H:] = \pi$$

$$i\partial_t \phi = i\int \frac{d^3\bar{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\bar{p}}}}\left(a_p(-i\omega)e^{-ip\cdot x} + a_p^\dagger(i\omega_{\bar{p}})e^{ip\cdot x}\right)$$

$$= \int \frac{d^3\bar{p}}{(2\pi)^3}\sqrt{\frac{\omega_{\bar{p}}}{2}}\underbrace{\left(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x}\right)}_{= \pi(x)} \checkmark$$

and

$$[\phi, :H:] = \int\frac{d^3\bar{k}}{(2\pi)^3}\int\frac{d^3\bar{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\bar{p}}}}\left[\omega_p a_p^\dagger a_p, a_k e^{-ik\cdot x} + a_k^\dagger e^{-ik\cdot x}\right]$$

$$= \int\frac{d^3\bar{k}}{(2\pi)^3}\int\frac{d^3\bar{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\bar{p}}}}\left\{\omega_{\bar{p}}[a_p^\dagger a_p, a_k]e^{-ik\cdot x} + \omega_{\bar{p}}[a_p^\dagger a_p, a_k^\dagger]e^{ik\cdot x}\right\}$$

$[ab,c] = a[b,c] + [a,c]b$ and $[a_p, a_k] = [a_p^\dagger, a_k^\dagger] = 0$

$$= \int\frac{d^3\bar{k}}{(2\pi)^3}\int\frac{d^3\bar{p}}{(2\pi)^3}\sqrt{\frac{\omega_{\bar{p}}}{2}}\left\{[a_p^\dagger, a_k]a_p e^{-ik\cdot x} + a_p^\dagger[a_p, a_k^\dagger]e^{ik\cdot x}\right\}$$

$[a_p^\dagger, a_k] = (2\pi)^3\delta^{(3)}(\bar{p}-\bar{k})$

$$= \int\frac{d^3p}{(2\pi)^3}\sqrt{\frac{\omega_{\bar{p}}}{2}}\left(-a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x}\right)$$

clearly equal to $i\partial_t\phi$! ✓

Similarly, one can show that

try it! (or see Tong page 35)

$$i\partial_t \pi = [\pi, :H:]$$
$$= (\bar{\nabla}^2 - m^2)\phi$$

Note that these Heisenberg equations of motion are completely consistent with the Klein-Gordon equation:

$$\left.\begin{aligned} \partial_t \phi &= \pi \\ \partial_t \pi &= (\bar{\nabla}^2 - m^2)\phi \end{aligned}\right\} \quad \begin{aligned} \partial_t(\partial_t \phi) &= \partial_t \pi \\ &= (\bar{\nabla}^2 - m^2)\phi \end{aligned}$$

$$\Rightarrow \underbrace{(\partial_t^2 - \bar{\nabla}^2 + m^2)\phi = 0}_{(\partial^2 + m^2)\phi = 0}$$

Our two pictures of the operators agree at a fixed time, say $t = 0$. The commutation relations we have introduced are equal-time commutation relations, so they apply equally to both pictures, provided we work at a fixed time. We will study how this changes later in the course when we introduce interactions.

Complex scalar field

$\boxed{\text{Schwarz 8.4, 9.1}}$

Tong 2.5

Before we look at quantising spinor fields, which embed spin-½ particles, let's look at a different kind of scalar field - the complex scalar field.

The complex scalar field can be treated as two independent real fields

two real fields

either $\phi(x) = \phi_1(x) + i\phi_2(x)$

or $\phi(x)$ and $\phi^*(x)$

two degrees of freedom

We quantise our complex scalar as

$$\phi(x) = \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p e^{-ip\cdot x} + b_p^+ e^{ip\cdot x} \right)$$

and

$$\phi^*(x) = \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p^+ e^{ip\cdot x} + b_p e^{-ip\cdot x} \right)$$

the full picture here requires coupling to a photon to see that the particle and antiparticle couple with opposite signs.

comparing these we see b_p^+ creates particles while a_p^+ creates antiparticles of the same mass

"particle travelling backwards in time"

An equivalent way to write this is

$$\phi_1(x) = \int \frac{d^3\bar{p}_1}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_{p,1} e^{-ip\cdot x} + a_{p,1}^+ e^{+ip\cdot x} \right)$$

$$\phi_2(x) = \int \frac{d^3\bar{p}_2}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_{p,2} e^{-ip\cdot x} + a_{p,2}^+ e^{ip\cdot x} \right)$$

this is like a polarisation vector

$$\Rightarrow \bar{\phi}(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} = \int \frac{d^3\bar{p}_j}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \sum_{j=1}^{2} \left(\bar{\epsilon}_j a_{p,j} e^{-ip\cdot x} + \bar{\epsilon}_j a_{p,j}^+ e^{ip\cdot x} \right)$$

$\bar{\epsilon}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\bar{\epsilon}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

A summary

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \hat\phi\, \partial^\mu \hat\phi - m^2 \hat\phi^2\right)$$

$$\Rightarrow (\partial^2 + m^2)\hat\phi^2 = 0 \quad \text{Klein-Gordon eq.}$$

Real scalar field

$$\hat\phi(\bar x, t) = \int \frac{d^3\bar k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \left(e^{-ik_\mu x^\mu} \hat a(\bar k) + e^{ik_\mu x^\mu} \hat a^+(\bar k)\right)$$

with conjugate momentum

$$\hat\pi(\bar x, t) = -i \int \frac{d^3 k}{(2\pi)^3} \sqrt{\frac{E_k}{2}} \left(e^{-ik_\mu x^\mu} \hat a(\bar k) - e^{ik_\mu x^\mu} \hat a^+(\bar k)\right)$$

satisfy

$$[\hat\phi(\bar x, t), \hat\pi(\bar y, t)] = i\delta^{(3)}(\bar x - \bar y) \qquad [\hat a(\bar k), \hat a^+(\bar p)] = (2\pi)^3 \delta^{(3)}(\bar k - \bar p)$$

- Single particle states are $|\bar k\rangle = \hat a^+(\bar k)|0\rangle$ and satisfy $\langle p|k\rangle = 2E_k (2\pi)^3 \delta^{(3)}(\bar p - \bar k)$

- Useful operators
  - Hamiltonian $\hat H = \frac{1}{2}\int d^3\bar x \left(\hat\pi^2 + \hat\phi(-\bar\nabla^2 + m^2)\hat\phi\right)$

    $\hat H = \int \frac{d^3\bar k}{(2\pi)^3} E_k\, \hat a^+(\bar k)\hat a(\bar k)$

    satisfies $[\hat H, \hat a^{(+)}] = E_k\, \hat a^{(+)}$ $\qquad \hat H|\bar k\rangle = E_k|\bar k\rangle$

  - Momentum operator $\hat P^i = -\int d^3\bar x\, \hat\pi \nabla^i \hat\phi$

    $\hat P^i = \int \frac{d^3\bar k}{(2\pi)^3} k^i\, \hat a^+(\bar k)\hat a(\bar k)$

    satisfies $[\hat P^i, \hat\phi] = i\bar\nabla^i \hat\phi$ $\qquad \hat P|\hat k\rangle = \bar k|\bar k\rangle$

  - Counting operator $\hat N = \int \frac{d^3\bar k}{(2\pi)^3} \hat a^+(\bar k)\hat a(\hat k)$

    satisfies $[\hat H, \hat N] = 0$ $\qquad \hat N|p_1, \dots, p_n\rangle = n|p_1, \dots, p_n\rangle$