Classical mechanics and quantisation

Schwarz 3.1 and 3.2
P+S 2.1 and
Tong Chap. 1

Let's start by considering a single nonrelativistic particle, and denote its position as $q(t)$

A classical state is specified by coordinates and momenta

$$\begin{tikzcd} CM \arrow[r] \arrow[d] & QM \arrow[d] \\ CFT \arrow[r] & QFT \end{tikzcd}$$

We introduce

- Lagrangian $L = \frac{1}{2}m\dot{q}^2 - V(q)$
- Canonical ("conjugate") momentum $p \equiv \frac{\partial L}{\partial \dot{q}} = m\dot{q}$
- Hamiltonian (defined via Legendre transformation)
  $$H \equiv p\dot{q} - L = \frac{1}{2}m\dot{q}^2 + V(q)$$
  H conserved iff $\frac{\partial L}{\partial t} = 0$

Dynamics are then described by

- Principle of least action (Hamilton's principle) $\delta S = 0$
  $\Rightarrow$ Lagrange equations $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q} \Rightarrow \underbrace{m\ddot{q} = -\frac{\partial V}{\partial q}}_{\text{Newton II}}$

- Hamilton's equations
  $$\left.\begin{aligned}\dot{q} &= \{q, H\}\\ \dot{p} &= \{p, H\}\end{aligned}\right\} \quad \{q_i, p_j\} = \delta_{ij}$$

N.B. $\{A, B\}_{q,p} \equiv \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}$

Quantisation is now "straightforward" in the Hamiltonian picture

- we treat position and momentum as operators $q, p \to \hat{q}, \hat{p}$
- and impose (equal-time) commutation relations $[\hat{q}, \hat{p}] = i$

N.B. $[\hat{q}, \hat{p}] = \hat{q}\hat{p} - \hat{p}\hat{q}$

A quantum state $|\psi\rangle$ specifies a single particle. Observables are operators in a Hilbert space.

Unfortunately the Hamiltonian picture is manifestly non-covariant. This makes the Lagrangian picture more useful for QFT (see 722).

Heisenberg picture

- operators evolve in time according to

$$\frac{d\hat{q}}{dt} = i\left[\hat{H}, \hat{q}\right] = \hat{p}$$

$$\frac{d\hat{p}}{dt} = i\left[\hat{H}, \hat{p}\right] = -\frac{\partial V}{\partial \hat{q}}$$

$$\frac{d\hat{O}}{dt} = i\left[\hat{H}, \hat{O}\right]$$

Recall: in the Schrödinger picture, states evolve with time and operators do not.

↑ equivalent !

- if $\frac{\partial \hat{H}}{\partial t} = 0$ then $\hat{O}(\hat{p}, \hat{q}; t) = e^{i\hat{H}t}\, O(\hat{p}, \hat{q}; t=0)\, e^{-i\hat{H}t}$

Now consider the generalisation of a single nonrelativistic particle to a <u>classical field</u>

- imagine a particle at each point on a grid $r_i(t)$

  ↑ labels grid points

  $\Rightarrow \quad L = \sum_i \frac{1}{2} m \dot{r}_i^2 - V(r_i)$ etc.

- obtain a field in the <u>continuum limit</u>

  $(i, t) \to x^\mu$ continuous spacetime

  $r_i(t) \to \phi(x^\mu)$ continuous scalar field

- introduce the Lagrange density

  $L \to \int d^3x \; \mathcal{L}(\phi, \partial_\mu \phi)$

  ↑ defined via $\left(\dot{q}_i(t),\, q_{i+1}(t) - q_i(t)\right) \to \frac{\partial \phi}{\partial x^\mu} = \partial_\mu \phi$

- equations of motion given by the Euler-Lagrange equations

$$\frac{\partial}{\partial x_\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)}\right) = \frac{\partial \mathcal{L}}{\partial \phi}$$

← Brief derivation on page 15 of P+S

We can also construct the Hamiltonian

- define the "canonical momentum" $\pi(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$
- and the Hamiltonian density $\mathcal{H} \equiv \pi(x)\dot{\phi}(x) - \mathcal{L}$

<u>Example</u>

Scalar field theory : a single, real-valued scalar defined at every point in spacetime

Defined by the Lagrange density $\mathcal{L} = \frac{1}{2} \underbrace{\partial_\mu \phi \cdot \partial^\mu \phi}_{\dot{\phi}^2 - \bar{\nabla}\phi \cdot \bar{\nabla}\phi} - \frac{1}{2} m^2 \phi^2$

Equations of motion are $(\partial^2 + m^2)\phi(x) = 0$

Euler-Lagrange equations

Klein-Gordon equation

Note: $\partial^2 \equiv \partial_\mu \partial^\mu = \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x_\mu} = \frac{\partial^2}{\partial t^2} - \bar{\nabla}^2$

$\partial_\mu = \left(\frac{\partial}{\partial x^0}, \bar{\nabla}\right) \qquad \partial^\mu = \left(\frac{\partial}{\partial x^0}, -\bar{\nabla}\right)$

Also: $\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} = \frac{\partial}{\partial(\partial^\mu \phi)}\left(\frac{1}{2} g_{\sigma\nu} \partial^\sigma \phi \, \partial^\nu \phi\right)$

$= \frac{1}{2} g_{\sigma\nu} \delta^{\mu\sigma} \partial^\nu \phi$

$+ \frac{1}{2} g_{\sigma\nu} \partial^\sigma \phi \, \delta^{\mu\nu} = \partial^\mu \phi$

Plane wave solutions to the equations of motion are

$$\phi(x) = A e^{-ip_\mu x^\mu} = A e^{-i(p^0 x^0 - \bar{p}\cdot\bar{x})} = A e^{-i(Et - \bar{p}\cdot\bar{x})}$$

$\Rightarrow \quad E^2 - |\bar{p}|^2 = m^2 \quad$ or $\quad p^2 = m^2$

relativistic dispersion relation

N.B. $(\partial_t^2 + |\bar{p}|^2 + m^2)\phi(\bar{p}, t) = 0$ is the E.of M. for an SHO with frequency $\omega_p = \sqrt{|\bar{p}|^2 + m^2}$

Canonical momentum in this case is

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$$

So the Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}\bar{\nabla}\phi \cdot \bar{\nabla}\phi + \frac{1}{2} m^2 \phi^2$$

Note Hamiltonian formulation non-covariant as $t, \bar{x}$ not treated equally.

Noether's theorem

$\boxed{\text{Schwarz } 3.3}$

P+S 2.2 and Tong 1.3

As we'll see symmetries are going to pop up everywhere in this course, so let's take a look at one of their manifestations in a classical context:

$\boxed{\text{Noether's theorem – every continuous symmetry of the Lagrangian generates a conserved current}}$

A continuous symmetry of the Lagrangian is a continuous transformation of the field(s) that leave the Lagrangian invariant.

In general:

$$\phi(x) \to \phi'(x) = \phi(x) + \alpha\,\delta\phi(x)$$

the continuous transformation that leads to

$$\mathcal{L}(\phi(x)) \to \mathcal{L}'(\phi'(x))$$

where $\mathcal{L}'(\phi'(x)) = \mathcal{L}(\phi(x)) + \alpha\,\partial_\mu J^\mu(\phi(x))$

is the condition that ensures invariance of the Lagrangian (up to a total four-divergence

Then the conserved current is

$$j^\mu(\phi(x)) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi - J^\mu$$

where conservation is expressed as

$$\partial_\mu j^\mu(\phi(x)) = 0.$$

Proof: $\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \alpha \delta \mathcal{L}$

$$\alpha \delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}(\alpha \delta\phi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\alpha \delta \phi)$$

$$= \alpha \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) + \alpha \underbrace{\left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\right]}_{=0 \text{ by E-L eqns.}}\delta\phi$$

$\frac{\partial j^0}{\partial t} + \bar{\nabla}\cdot\bar{j} = 0$ see D.Tong p. 13

Note:

- a conserved current $\Rightarrow$ a conserved charge

$$Q = \int_{R^3} d^3x \, j^0$$

$$\frac{dQ}{dt} = \int_{R^3} d^3\bar{x} \frac{dj^0}{dt} = -\int_{R^3} d^3\bar{x}\, \bar{\nabla}\cdot\bar{j} = 0$$

for $j \to 0$ as $|\bar{x}| \to \infty$

- but a conserved current is a stronger constraint, because it implies charge is conserved locally.

consider charge in a finite volume and apply Stokes' law

$$\frac{dQ_V}{dt} = \int_V d^3\bar{x} \frac{dj^0}{dt} = -\int_V d^3\bar{x}\, \bar{\nabla}\cdot\bar{j} = -\int_A \bar{j}\cdot d\bar{S}$$

<u>Example</u>

Consider massless free scalar field theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2$$

strictly speaking, this would be a global, not local, symmetry

Invariant under an analogue of <u>chiral symmetry</u> $\phi(x) \to \phi'(x) = \phi(x) + \alpha$

relevant to QCD

α a constant

This symmetry is broken by the presence of a mass term

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m^2\phi^2$$

$$m^2\phi^2 \to m^2\phi^2 + 2\alpha m^2 \phi + m^2 \alpha^2$$

Example

Consider the infinitesimal spacetime translation $x^\mu \to x^\mu - a^\mu$

We can cast this as a field transformation

$$\phi(x) \to \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi(x)$$

This transformation behavior is true for any scalar

$$\Rightarrow \mathcal{L}(\phi(x)) \to \mathcal{L}(\phi(x+a)) = \mathcal{L}(\phi(x)) + a^\mu \partial_\mu \mathcal{L}(\phi(x))$$
$$= \mathcal{L}(\phi(x)) + a^\nu \partial_\mu \underbrace{(\delta^\mu{}_\nu \mathcal{L}(\phi(x)))}_{\text{"}J^\mu\text{"}}$$

"J^μ" ie. our 4-divergence from before

In fact four conserved currents

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu{}_\nu \mathcal{L}$$

This is the <u>energy-momentum</u> (or stress-energy) <u>tensor</u>

- $H = \int T^{00} d^3\bar{x} = \int \mathcal{H} \, d^3\bar{x}$ is conserved charge

  invariance under time translation generates energy conservation

- $P^i = \int T^{0i} d^3\bar{x} = -\int \hat{\pi}\, \partial_i \phi \, d^3\bar{x}$

  invariance under spatial translations generates momentum conservation

Do not confuse physical momentum carried by the field (P^i) with the canonical momentum (π)

strictly this is the momentum density

Note that we must carefully distinguish

- spacetime symmetries – symmetries under Lorentz transformations, which act directly on x^μ (and on the field defined at x^μ, as a result)

continuous e.g. rotations | discrete e.g. parity

- internal symmetries – symmetries of the fields themselves

global – same at all spacetime points | local – depends on the spacetime point

The classic example is the complex scalar field (which can be treated as two copies of a real scalar field, either as $\phi(x)$ and $\phi^*(x)$ or as $\phi_R(x) + i\phi_I(x)$), which has Lagrangian

$$\mathcal{L} = |\partial_\mu \phi|^2 - m^2|\phi|^2 = \partial^\mu\phi\,\underbrace{\partial_\mu\phi^* - m^2\phi\phi^*}_{\text{equations of motion: } (\partial^2+m^2)\phi = 0}$$

that is invariant under

$$\phi(x) \to \phi'(x) = e^{-i\alpha}\phi \qquad \text{for any } \alpha \in \mathbb{R}$$

N.B. $\dfrac{\delta\phi}{\delta\alpha} = -i\phi \qquad \dfrac{\delta\phi^*}{\delta\alpha} = i\phi^*$

$$\Rightarrow j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\frac{\delta\phi}{\delta\alpha} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\frac{\delta\phi^*}{\delta\alpha} = -i(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi)$$

$$\Rightarrow \partial_\mu j^\mu = -i(\phi\partial^2\phi^* - \phi^*\partial^2\phi)$$

by equations of motion

$$= -i(\phi(-m^2\phi^*) - \phi^*(-m^2\phi))$$

$$= 0$$

Plane waves and oscillators

"The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction." Sidney Coleman

We all are familiar with the classical simple harmonic oscillator (SHO)

$$m\frac{d^2x}{dt^2} + kx = 0 \qquad \omega = \sqrt{\frac{k}{m}}$$

or

$$\ddot{x} + \omega x = 0$$

with solution

$$x(t) = c_1 e^{i\omega t} + c_2 e^{-i\omega t}$$

The Hamiltonian for this system is

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

which we quantise by putting hats on everything

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 \qquad \text{with } [\hat{x}, \hat{p}] = i$$

To analyse this, we introduce the <u>raising</u> and <u>lowering operators</u> or "ladder operators"

$$\hat{a}_{\pm} = \sqrt{\frac{m\omega}{2}}\left(\hat{x} \pm \frac{i\hat{p}}{m\omega}\right) \qquad \text{with } [a_+, a_-] = 1$$

and write our Hamiltonian as

$$\hat{H} = \omega\left(\hat{a}_-\hat{a}_+ + \frac{1}{2}\right)$$

this is the <u>number operator</u> $\hat{N} = \hat{a}_-\hat{a}_+$

The eigenstates of the number operator are

$$\hat{N}|n\rangle = n|n\rangle$$

and satisfy

$$a_-|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$a_+|n\rangle = \sqrt{n}\,|n-1\rangle$$

← this is the raising operator so "$\hat{a}_-$" is a bit confusing and it is often called $\hat{a}^+$

In the Heisenberg picture, the operators evolve as

$$\dot{\hat{a}}_+ = i[\hat{H}, \hat{a}_+]$$

$$= i\left[\omega\left(\hat{a}_-\hat{a}_+ + \tfrac{1}{2}\right), \hat{a}_+\right]$$

$$= i\omega\left(\hat{a}_-\hat{a}_+\hat{a}_+ - \hat{a}_+\hat{a}_-\hat{a}_+\right)$$

$$= i\omega\left([\hat{a}_-, \hat{a}_+]\hat{a}_+\right)$$

$$= i\omega\hat{a}_+$$

And this has solution $\hat{a}(t) = e^{-i\omega t}\hat{a}_-(0)$

The textbook proceeds to go ahead from here with "second quantisation", but we're going to detour a little before coming back to this. Schwarz's presentation does at least make it clear why it is called the "second quantisation".

Not my favourite name.