General Physics 1–Honors (PHYS 101H):
Problem Set 7–Solutions

Chris Monahan
Department of Physics, William & Mary

Overview

The written Problem Sets will help you gain experience with how to present your solutions to university-level physics problems. This will be necessary for your midterm and final exams, as well as future courses thoughout your undergraduate career. Present your solutions legibly and as logically as you can. What this means in practice is the following:

1. Write down what quantities you know.
2. Write down the relevant equations.
3. When carrying out manipulations or substituting values into equations, try to write each equality on separate lines. For example:

$$\begin{aligned} x(t_0) &= \frac{1}{2}at_0^2 + v_0t_0 + x_0 \\ &= \frac{1}{2} \cdot 9.8 \cdot 1.0^2 + 1.5 \cdot 1.0 + 0.7 \\ &= \boxed{7.1\,\mathrm{m}}. \end{aligned}$$

4. Circle or underline your final answers to identify them clearly (see the equation above)..

I will post solutions that will also provide one possible model for how to present solutions.

Some hints:

1. Only substitute values at the end of your calculation and try to carry out all manipulations symbolically.
2. Double check the order of magnitude of your answer.
3. Double check the units of your answer.
4. Double check the number of significant figures of your answer (do not give more significant figures than the question provides for physical quantities).

This Problem Set, in particular, will provide practice in applying conditions of static equilibrium and studying orbital motion.

This Problem Set is worth 50 points; there are three questions in this Problem Set.

Instructions

Read these instructions carefully. You must submit your Problem Set as a **single PDF** file (it is best to use an app like Adobe Scan to make your solutions legible), with the file name `lastname_hwXX.pdf` (replace `lastname` with your last name and `XX` with the problem set number). If you do not submit your Problem Set according to these instructions, you will be deducted five points.

Question 1 **10pts**

A ladder leans against a vertical wall, making an angle θ with the horizontal. The coefficient of static friction between the ladder and both the floor and the wall is $\mu \leq 1$. What is the minimum value of the angle θ (expressed in terms of μ) for which it is possible for the ladder not to fall? Assume that the maximum value of θ occurs when the friction forces take on their maximum values.

Solution 1

Figure 1 illustrates the ladder system in question 1.

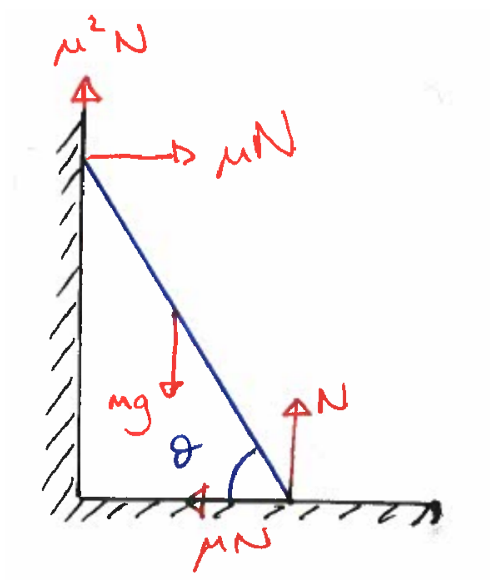

Figure 1: Diagram for Solution 1.

There are several ways to approach this problem, but all use some combination of our definition of equilibrium: the net force and net torque on the system are both zero.

The simplest approach, algebraically, is to study the torques around the centre of the ladder. Let's define anticlockwise rotations as positive.

First we identify that, in the limiting case, the friction due to the floor is μN. The normal force due to the wall is the only other horizontal force, so equilibrium requires that the normal force from the wall is μN. This means that the friction force on the wall is $\mu \cdot \mu N = \mu^2 N$.

Now let's look at the torques, which must sum to zero for equilibrium. They are given by

$$\mu^2 N \cos\theta \frac{\ell}{2} + \mu N \sin\theta \frac{\ell}{2} - N \cos\theta \frac{\ell}{2} + \mu N \sin\theta \frac{\ell}{2} = 0.$$

We can rearrange this as

$$2\mu \sin\theta = \cos\theta (1 - \mu^2),$$

or

$$\boxed{\tan\theta = \frac{1-\mu^2}{2\mu}}.$$

This condition specifies the minimum value of θ.

Question 2 **20pts**

Two children are balanced on a horizontal, stationary seesaw. One child, Alpha, sits at the furthest righthand end of the seesaw, a distance a from the middle of the seesaw. The second child, Zelda, sits at a distance b to the left of the middle of the seesaw.

(a) If Alpha has a mass m_A, find Zelda's mass.

(b) Suppose that Alpha's younger sibling, Alexa, joins Zelda on the lefthand side of the seesaw, and sits at the far left end. Assume that the Alexa's mass is half that of Alpha and find Alexa's initial linear acceleration.

(c) Find the position where Zelda must sit to ensure that all three children can balance the seesaw horizontally at rest.

Solution 2

(a) The seesaw is initially in equilibrium, so the net torque about any axis is zero. Let's choose the middle of the seesaw as the point around which we calculate the torques, in which case we have

$$\tau_A - \tau_Z = 0.$$

Now $\tau_A = m_A g \cdot a$ and $\tau_Z = m_Z g \cdot b$, so we have

$$m_A g a = m_Z g b,$$

or

$$\boxed{m_Z = \frac{a m_A}{b}}.$$

(b) The net torque is now

$$\tau_a + \tau_Z - \tau_A = I\alpha,$$

where I is the moment of inertia of the system and α is the angular acceleration. Thus

$$\frac{m_A}{2} g \cdot a + m_Z g \cdot b - m_A g \cdot a = I\alpha.$$

Plugging in our value for m_Z gives

$$\begin{aligned} I\alpha &= m_Z g b - \frac{a g m_A}{2} \\ &= \frac{a m_A}{b} \cdot g b - \frac{m_A g a}{2} \\ &= \frac{a g m_A}{2}. \end{aligned}$$

The moment of inertia of the system is

$$I = \sum_{i=1}^{3} m_i r_i^2,$$

which is

$$\begin{aligned}I &= \frac{m_A}{2}a^2 + m_Z b^2 + m_A a^2 \\ &= \frac{3m_A a^2}{2} + \frac{a m_A}{b} b^2 \\ &= a m_A \left(\frac{3a}{2} + b\right).\end{aligned}$$

Then the angular acceleration is

$$\begin{aligned}\alpha &= \frac{\tau}{I} \\ &= \frac{a g m_A/2}{a m_A\,(3a/2 + b)} \\ &= \frac{g}{2\,(3a/2 + b)} \\ &= \frac{g}{3a + 2b}.\end{aligned}$$

The linear acceleration is tangential, and related to the angular acceleration via

$$\alpha = \frac{a_t}{r},$$

so the linear acceleration of Alexa (at distance $r = a$) is

$$a_t = \alpha \cdot r = \boxed{\frac{ag}{3a + 2b}}.$$

(c) For all three children to be at rest, we require

$$\tau_a + \tau_Z - \tau_A = 0,$$

or

$$\frac{m_A}{2} g \cdot a + m_Z g \cdot c - m_A g \cdot a = 0,$$

where c is Zelda's new position. Plugging in our expression for m_Z gives

$$\frac{m_A}{2} g \cdot a + \frac{a m_A}{b} g \cdot c - m_A g \cdot a = 0,$$

or

$$-\frac{a g m_A}{2} + \frac{a m_A g c}{b} = 0,$$

Thus

$$\boxed{c = \frac{b}{2}}.$$

Question 3 **20pts**

(a) A satellite with mass m is in a circular orbit with radius R around a planet with mass M, which is assumed to be fixed in place. What is the total energy of the statellite, expressed in terms of G, m, M, and R?

(b) Assume now that the satellite has been given a kick so that its orbit is an ellipse, with the closest approach to M is R and the furthest distance is nR (n is a given numerical factor). This is illustrated in figure 2. What is the total energy of the satellite now?

(c) Check and comment on the $n \to 1$, $n \to \infty$, and $n \to 0$ limits.

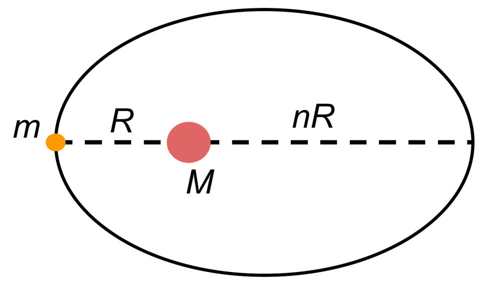

Figure 2: Diagram for Question 3.

Solution 3

(a) The potential energy at radius R is

$$E_P = -\frac{GMm}{R}.$$

To find the kinetic energy, we need to apply Newton's second law, which gives

$$\frac{GMm}{R^2} = \frac{mv^2}{R}.$$

Thus

$$v^2 = \frac{GM}{R}.$$

The kinetic energy is therefore

$$E_K = \frac{mv^2}{2} = \frac{GMm}{2R}.$$

The total energy is

$$E_{\text{TOT}} = E_K + E_P = \boxed{-\frac{GMm}{2R}}.$$

The fact that the total energy is negative indicates that the system is *bound*, or, in other words, energy would have to be put into the system (work done on the system) to move the satellite out to infinity.

(b) Let v_1 be the speed at R and v_n the speed at nR. Conservation of angular momentum tells us that

$$mv_1R = mv_n(nR),$$

or

$$v_n = \frac{v_1}{n}.$$

Conservation of energy therefore requires

$$\begin{aligned}\frac{mv_1^2}{2} - \frac{GMm}{R} &= \frac{mv_n^2}{2} - \frac{GMm}{nR} \\ &= \frac{m}{2}\left(\frac{v_1}{n}\right)^2 - \frac{GMm}{nR} \\ &= \frac{mv_1^2}{2n^2} - \frac{GMm}{nR}.\end{aligned}$$

This can be rearranged as

$$\frac{mv_1^2}{2} - \frac{mv_1^2}{2n^2} = \frac{GMm}{R} - \frac{GMm}{nR},$$

or

$$\frac{mv_1^2}{2}\left(1 - \frac{1}{n^2}\right) = \frac{GMm}{R}\left(1 - \frac{1}{n}\right).$$

Using

$$\left(1 - \frac{1}{n^2}\right) = \left(1 - \frac{1}{n}\right)\left(1 + \frac{1}{n}\right),$$

this becomes

$$\frac{mv_1^2}{2}\left(1 + \frac{1}{n}\right) = \frac{GMm}{R},$$

or

$$\frac{mv_1^2}{2} = \frac{n}{n+1}\frac{GMm}{R}.$$

This is exactly the kinetic energy at R!

Thus the total energy is

$$\begin{aligned} E_{\text{TOT}} &= E_K + E_P \\ &= \frac{n}{n+1}\frac{GMm}{R} - \frac{GMm}{R} \\ &= \boxed{-\frac{GMm}{(n+1)R}}. \end{aligned}$$

(c) In the limit $n \to 1$, the total energy becomes

$$E_{\text{TOT}} = -\frac{GMm}{2R},$$

in agreement with the spherical orbit of part (a) (that is, exactly what we expect in this limit!).

In the limit $n \to \infty$, we have

$$E_{\text{TOT}} \to 0.$$

This is again in line with what we would expect, because as the satellite moves towards infinity, it will be moving arbitrarily slowly and be an almost infinite distance from M. So both its kinetic energy and its potential energy will tend to zero.

For $n \to 0$, the total energy is

$$E_{\text{TOT}} \to -\frac{GMm}{R},$$

which is equal to the gravitational potential energy at R. This makes sense, because this limit corresponds to essentially "dropping" the satellite vertically down towards the mass M, and, in this case, the satellite is essentially at rest at the "top" of its motion. Thus the only energy at the extreme of its very very thin elliptical orbit is potential energy.