General Physics 1–Honors (PHYS 101H): Problem Set 6–Solutions

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Overview

The written Problem Sets will help you gain experience with how to present your solutions to university-level physics problems. This will be necessary for your midterm and final exams, as well as future courses thoughout your undergraduate career. Present your solutions legibly and as logically as you can. What this means in practice is the following:

1. Write down what quantities you know.
2. Write down the relevant equations.
3. When carrying out manipulations or substituting values into equations, try to write each equality on separate lines. For example:

$$\begin{aligned} x(t_0) &= \frac{1}{2}at_0^2 + v_0t_0 + x_0 \\ &= \frac{1}{2}\cdot 9.8 \cdot 1.0^2 + 1.5 \cdot 1.0 + 0.7 \\ &= \boxed{7.1\,\mathrm{m}}. \end{aligned}$$

4. Circle or underline your final answers to identify them clearly (see the equation above)..

I will post solutions that will also provide one possible model for how to present solutions.

Some hints:

1. Only substitute values at the end of your calculation and try to carry out all manipulations symbolically.
2. Double check the order of magnitude of your answer.
3. Double check the units of your answer.
4. Double check the number of significant figures of your answer (do not give more significant figures than the question provides for physical quantities).

This Problem Set, in particular, will provide practice in moments of inertia, torque, and angular momentum in two dimensions.

This Problem Set is worth 50 points; there are three questions in this Problem Set.

Instructions

Read these instructions carefully. You must submit your Problem Set as a **single PDF** file (it is best to use an app like Adobe Scan to make your solutions legible), with the file name `lastname_hwXX.pdf` (replace `lastname` with your last name and `XX` with the problem set number). If you do not submit your Problem Set according to these instructions, you will be deducted five points.

Question 1 **10pts**

A massless stick lies on a table, with a length a hanging over the edge and a length b on the table, as shown in figure 1. A ball with mass m lies on the stick at its right end and a second ball with mass m is dropped above the left end of the stick. This second ball hits the stick with speed v_0. Assuming that all collisions are elastic, what are the velocities of the two balls right after the collision?

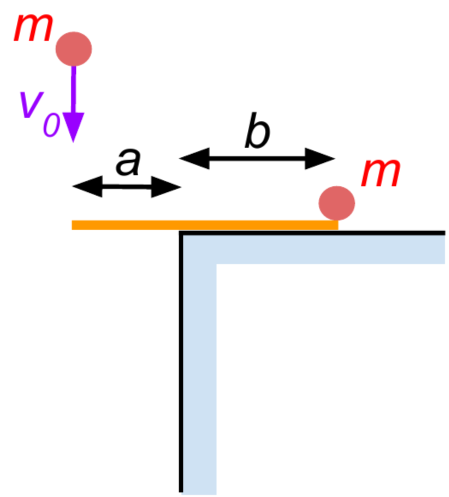

Figure 1: Diagram for Question 1.

Solution 1

The collision is elastic, so kinetic energy is conserved in the collision. Moreover, angular momentum is also conserved in the process, because the table provides no net torque on the stick (the only force applied by the table on the stick during the collision is at the point of rotation, the corner of the table).

We will define the velocities to be v_1 upward for the ball at the righthand end of the stick (the ball on the table) and v_2 downwards for the ball that strikes the stick at its lefthand end.

Conservation of energy tells us that

$$\frac{mv_0^2}{2} = \frac{mv_1^2}{2} + \frac{mv_2^2}{2},$$

or

$$v_0^2 = v_1^2 + v_2^2.$$

Conservation of angular momentum means

$$mv_0a = mv_1b + mv_2a.$$

Thus

$$v_2 = v_0 - \frac{bv_1}{a}.$$

Putting these together, we have

$$v_0^2 = v_1^2 + \left(v_0 - \frac{bv_1}{a}\right)^2,$$

or

$$v_0^2 = v_1^2 + v_0^2 - \frac{2bv_0v_1}{a} + \frac{b^2v_1^2}{a^2}.$$

Rearranging this gives

$$0 = \left[-\frac{2bv_0}{a} + \left(1 + \frac{b^2}{a^2}\right) v_1\right] v_1.$$

The nontrivial solution is

$$\boxed{v_1 = v_0 \frac{2ab}{a^2 + b^2}.}$$

Plugging this into our conservation of angular momentum equations leads us to

$$\begin{aligned} v_2 &= v_0 - \frac{b}{a}\frac{2v_0ab}{a^2+b^2} \\ &= v_0\left(1 - \frac{2b^2}{a^2+b^2}\right) \\ &= \boxed{v_0 \frac{a^2-b^2}{a^2+b^2}}. \end{aligned}$$

Question 2 **20pts**

A uniform disc with mass m and radius R lies in a vertical plane and is pivoted at its center. A stick with length ℓ and uniform mass density, λ, is glued tangentially at its top end to the disc. This setup is shown in figure 2.

(a) Show that the moment of inertia is

$$I = \frac{mR^2}{2} + \frac{\lambda \ell^3}{3} + \lambda \ell R^2.$$

(b) Assuming that the stick is held vertically and then released, what is the initial angular acceleration of the system?

(c) For what value of ℓ (expressed in terms of m, R, and λ) is this angular acceleration a maximum?

Solution 2

(a) The total moment of inertia is given by the sum of moments of inertia of the disc and the stick. The moment of inertia of the stick is given by the parallel axis theorem, which gives

$$I_{\text{stick}} = I_{\text{CM}} + md^2,$$

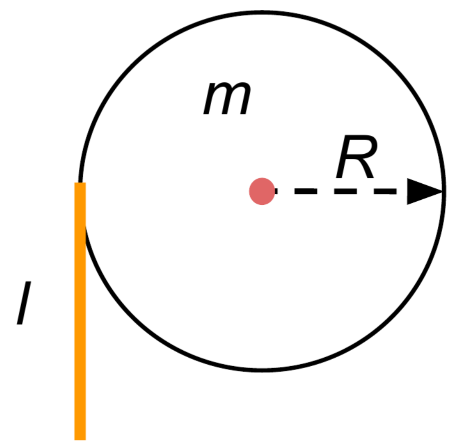

Figure 2: Diagram for Question 2.

where d is the distance from the centre of mass of the stick to the axis of rotation, which is given by

$$d^2 = \left(\frac{\ell}{2}\right)^2 + R^2.$$

Thus

$$I_{\text{stick}} = \frac{\lambda \ell^3}{12} + (\lambda \ell)\left[\left(\frac{\ell}{2}\right)^2 + R^2\right].$$

The total moment of inertia is therefore

$$\begin{aligned} I &= I_{\text{disc}} + I_{\text{stick}} \\ &= \frac{mR^2}{2} + \frac{\lambda \ell^3}{12} + (\lambda \ell)\left[\left(\frac{\ell}{2}\right)^2 + R^2\right] \\ &= \boxed{\frac{mR^2}{2} + \frac{\lambda \ell^3}{3} + \lambda \ell R^2}, \end{aligned}$$

as required.

(b) The angular acceleration is related to the moment of inertia via

$$\tau = I\alpha,$$

where τ is the torque. In this case the torque is due to the weight of the stick acting vertically downwards, which is

$$\tau = \lambda \ell g R.$$

Thus the angular acceleration is

$$\begin{aligned} \alpha &= \frac{\tau}{I} \\ &= \frac{\lambda \ell g R}{\frac{mR^2}{2} + \frac{\lambda \ell^3}{3} + \lambda \ell R^2} \\ &= \boxed{\frac{6\lambda \ell g R}{3mR^2 + 2\lambda \ell^3 + 6\lambda \ell R^2}}. \end{aligned}$$

(c) To find the maximum, we need to solve

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\ell} = 0.$$

The derivative is given by the quotient rule

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\ell} = \frac{6\lambda gR \cdot (3mR^2 + 2\lambda\ell^3 + 6\lambda\ell R^2) - 6\lambda\ell gR(6\lambda\ell^2 + 6\lambda R^2)}{(3mR^2 + 2\lambda\ell^3 + 6\lambda\ell R^2)^2}.$$

For this to be equal to zero, we require that the numerator is equal to zero. We can simplify the numerator as

$$\begin{aligned} &6\lambda gR \cdot (3mR^2 + 2\lambda\ell^3 + 6\lambda\ell R^2) - 6\lambda\ell gR(6\lambda\ell^2 + 6\lambda R^2) \\ &\quad = 6\lambda gR \left[3mR^2 + 2\lambda\ell^3 + 6\lambda\ell R^2 - \lambda\ell(6\lambda\ell^2 + 6\lambda R^2)\right] \\ &\quad = 6g\lambda R \left(3mR^2 - 4\lambda\ell^3\right). \end{aligned}$$

The nontrivial solution for this is

$$3mR^2 - 4\lambda\ell^3 = 0,$$

which means the length that maximises the angular acceleration is

$$\boxed{\ell = \left(\frac{3mR^2}{4\lambda}\right)^{1/3}}.$$

Question 3 **20pts**

Two uniform solid cylinders are placed on a plane inclined at an angle θ with respect to the horizontal. Both cylinders have mass m, but one radius is twice the other. A massless string connects the axle of the larger cylinder to the rim of the smaller cylinder, as illustrated in figure 3 The cylinders are released from rest.

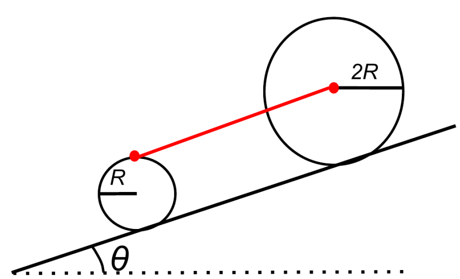

Figure 3: Diagram for Question 3.

(a) Assuming that the cylinders roll without slipping down the plane, what are their accelerations?

(b) If the coefficient of static friction between both cylinders and the plane is $\mu_S = 1.2$, what is the largest angle θ for which no slipping occurs between either cylinder and the plane?

Solution 3

(a) Let's work in a coordinate frame that is parallel and perpendicular to the slope. Each cylinder experiences the tension in the string, the friction force due to the plane, and a component of gravity along the surface. The string means that the linear acceleration of the larger cylinder is equal to the linear acceleration of the top of the rim of the smaller cylinder. The acceleration of the top of the cylinder is given by

$$a_{\text{rim}}^{(S)} = a_{\text{centre}}^{(S)} + \alpha R,$$

but since there is no slipping, $a_{\text{centre}}^{(S)} = \alpha R$. Thus

$$a_{\text{rim}}^{(S)} = 2\alpha R.$$

Therefore the linear acceleration of the larger cylinder is

$$a_{\text{centre}}^{(L)} = a_{\text{rim}}^{(S)} = 2\alpha R,$$

whereas the linear acceleration of the centre of the smaller cylinder is

$$a_{\text{centre}}^{(S)} = \alpha R.$$

Now must apply Newton's second law to both cylinders. For the small cylinder:

$$\begin{aligned} mg\sin\theta - F_S - T &= m a_{\text{centre}}^{(S)}, \\ F_S R - TR &= \frac{mR^2}{2}\alpha. \end{aligned}$$

For the larger cylinder,

$$\begin{aligned} T + mg\sin\theta - F_L &= m a_{\text{centre}}^{(L)}, \\ F_L \cdot 2R &= \frac{m(2R)^2}{2}\alpha. \end{aligned}$$

These are four equations, for four unknowns. To make progress, and connect the equations further, we can use $\alpha = a/R$, and simplify the equations a little to find

$$\begin{aligned} mg\sin\theta - F_S - T &= m a_{\text{centre}}^{(S)}, \\ F_S - T &= \frac{m a_{\text{centre}}^{(S)}}{2}, \\ T + mg\sin\theta - F_L &= m a_{\text{centre}}^{(L)}, \\ F_L &= m\frac{a_{\text{centre}}^{(L)}}{2}. \end{aligned}$$

We note that

$$a_{\text{centre}}^{(L)} = 2a_{\text{centre}}^{(S)} \equiv 2a,$$

so these equations become

$$\begin{aligned} mg\sin\theta - F_S - T &= ma, \\ F_S - T &= \frac{ma}{2}, \\ T + mg\sin\theta - F_L &= 2ma, \\ F_L &= ma. \end{aligned}$$

Adding the first and third equations gives

$$mg\sin\theta - 2T = \frac{3ma}{2}.$$

Adding the second and fourth gives

$$T + mg\sin\theta = 3ma.$$

Subtracting the first from the second leads us to

$$3T = \frac{3ma}{2},$$

and adding twice the second to the first gives

$$3mg\sin\theta = \frac{15ma}{2}.$$

Thus

$$a = \frac{2}{5}g\sin\theta,$$

and therefore

$$T = \frac{1}{5}mg\sin\theta.$$

The accelerations are therefore

$$\boxed{a^{(L)}_{\text{centre}} = \frac{4}{5}g\sin\theta}, \qquad \text{and} \qquad \boxed{a^{(S)}_{\text{centre}} = \frac{2}{5}g\sin\theta}.$$

(b) Plugging in our results for the tension and the acceleration, we have

$$F_S = F_L = \frac{2}{5}mg\sin\theta.$$

For the cylinders not to slip, we require

$$F_S \leq \mu N = \mu mg\cos\theta.$$

Thus

$$\frac{2}{5}mg\sin\theta \leq \mu mg\cos\theta,$$

or

$$\tan\theta \leq \frac{5\mu}{2}.$$

This corresponds to an angle of

$$\theta \leq \arctan\left(\frac{5\mu}{2}\right) \approx \boxed{71.6^\circ}.$$