General Physics 1–Honors (PHYS 101H): Problem Set 5–Solutions

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Overview

The written Problem Sets will help you gain experience with how to present your solutions to university-level physics problems. This will be necessary for your midterm and final exams, as well as future courses thoughout your undergraduate career. Present your solutions legibly and as logically as you can. What this means in practice is the following:

1. Write down what quantities you know.
2. Write down the relevant equations.
3. When carrying out manipulations or substituting values into equations, try to write each equality on separate lines. For example:

$$\begin{aligned} x(t_0) &= \frac{1}{2}at_0^2 + v_0t_0 + x_0 \\ &= \frac{1}{2}\cdot 9.8 \cdot 1.0^2 + 1.5 \cdot 1.0 + 0.7 \\ &= \boxed{7.1\,\mathrm{m}}. \end{aligned}$$

4. Circle or underline your final answers to identify them clearly (see the equation above)..

I will post solutions that will also provide one possible model for how to present solutions.

Some hints:

1. Only substitute values at the end of your calculation and try to carry out all manipulations symbolically.
2. Double check the order of magnitude of your answer.
3. Double check the units of your answer.
4. Double check the number of significant figures of your answer (do not give more significant figures than the question provides for physical quantities).

This Problem Set, in particular, will provide practice in applying conservation of momentum in one dimension and calculating the centre of mass for three different objects.

This Problem Set is worth 50 points; there are three questions in this Problem Set.

Instructions

Read these instructions carefully. You must submit your Problem Set as a **single PDF** file (it is best to use an app like Adobe Scan to make your solutions legible), with the file name `lastname_hwXX.pdf` (replace `lastname` with your last name and `XX` with the problem set number). If you do not submit your Problem Set according to these instructions, you will be deducted five points.

Question 1 **15pts**

You sit on a crate on frictionless ice. You are happy. The combined mass of you and the crate is m.

(a) You are now given a block of mass m to hold. Since you don't like this block, you throw it away, with relative horizontal speed v_0. What is your resulting speed (assuming you are still sitting on the crate)?

(b) The person that gave you the block is disappointed that you threw their block away, so when you come to rest (still on the ice), they give you another block of mass m. This time you dislike it so much you break it in half, then throw away the first half, wait a moment and then throw away the second half. What is your resulting speed? Assume that you throw each piece horizontally and the relative speed each time is v_0.

(c) You are starting to exasperate the person that gave you the blocks, but they persist and give you another block of mass m. You don't understand why. At this point, you are so mad that you break the block into n pieces of equal mass and throw each one away individually (again at relative speed v_0). Find your resulting speed after you have thrown all pieces away (your result should take the form $f(n)v_0$, where $f(n)$ is the function you are trying to determine).

Solution 1

(a) Your initial momentum is zero, so the final momentum, after you throw the block, must also be zero (by conservation of momentum). Let the speeds after you throw the block away, relative to the ground, be v_b for the block and v_c for you and the crate. Conservation of momentum gives us

$$mv_b = mv_c,$$

so $v_c = v_b$. The **relative speed** of you and the block is given in the question and must be $v_0 = v_b + v_c$. Solving these two equations gives your final speed as

$$v_c = v_0 - v_c$$

or

$$\boxed{v_c = \frac{v_0}{2}}.$$

(b) After you throw the first block, conservation of momentum tells us that

$$\frac{m}{2}v_b' = \frac{3m}{2}v_c'$$

where $v_0 = v_b' + v_c'$. This leads us to

$$v_c' = v_0 - v_b' = v_0 - 3v_c'.$$

Thus $v_c' = v_0/4$.

Now you throw the second part of the block and conservation of momentum this time means

$$\frac{m}{2}v_b = mv_c.$$

Note that we don't have to consider the momentum of the first piece (or your own initial momentum arising from throwing that first piece), because they will simply cancel out in this equation. Using $v_0 = v_b + v_c$ again, we find

$$v_c = v_0 - v_b = v_0 - 2v_c.$$

Thus the second throw gives you an extra speed of $v_0/3$.

Your final speed is the sum of these two speeds, $v_0/4 = v_0/3$. Thus your final speed is

$$\boxed{v_c = \frac{7v_0}{12}}.$$

(c) Notice that the final speed in part (b) could be written as

$$v_0 \left(\frac{1}{4} + \frac{1}{3} \right).$$

If you repeat this calculation for three pieces, you will find

$$v_0 \left(\frac{1}{6} + \frac{1}{5} + \frac{1}{4} \right).$$

The patten here is that you can always write the expression as

$$v_0 \left(\frac{1}{2n} + \cdots + \frac{1}{n+1} \right),$$

for n pieces. Thus, for n pieces, the final speed is

$$\boxed{v_c = f(n)v_0 = v_0 \sum_{n+1}^{2n} \frac{1}{n}}, \qquad \text{or} \qquad f(n) = \sum_{n+1}^{2n} \frac{1}{n}.$$

Question 2 **15pts**

A hose shoots a stream of water vertically, with an initial speed of v_0 and mass flow rate of R (measured in kg/s).

(a) What is the maximum height that the water will reach?

A horizontal board is now placed a very small distance above the hose and released.

(b) Find the mass of the board that is necessary for the board to hover at this height. Assume that the water bounces of the board essentially sideways.

(c) If you halve the mass of the board (by breaking it in half, for example), how high above the hose should you place the board so that it hovers in place?

The stream of water is now replaced by a stream of marbles that strike the original board elastically. Assume that the marbles arrive at the board almost continuously (with the same mass rate, R, as the water) and do not interfere with each other once they bounce off the board.

(d) Find the mass of the board that is necessary for the board to hover at a very small height.

Solution 2

(a) To solve this, we can use conservation of energy for a small mass of water $\mathrm{d}m$. The initial kinetic energy is converted to potential energy, which is at a maximum at the maximum height. Thus

$$\frac{\mathrm{d}m \cdot v_0^2}{2} = \mathrm{d}m \cdot gh.$$

Solving this for h gives

$$\boxed{h = \frac{v_0^2}{2g}}.$$

Note that the same result can be obtained by using the flow rate R to find the **power** (the time rate of change of the energy), since effectively all this does is replace $\mathrm{d}m$ with R in the above equation.

(b) The magnitue of the force pushing the board upwards (from the water) is equal to the magnitude of the force from the water on the board (Newton's third law). This force, from the board, causes the rate of change of momentum

$$F = \frac{\mathrm{d}p}{\mathrm{d}t}.$$

The magnitude of the change of momentum for a small mass of water is

$$\mathrm{d}p = \mathrm{d}m \cdot v_0,$$

and therefore the force is

$$F = \frac{(v_0 \mathrm{d}m)}{\mathrm{d}t} = v_0 \frac{\mathrm{d}m}{\mathrm{d}t} = v_0 R.$$

The board will hover in place if this force (which is equal to the force of the water acting on the board) is equal to the weight of the board, so

$$mg = v_0 R,$$

or

$$\boxed{m = \frac{v_0 R}{g}}.$$

(c) For a board of half the mass, we need half of the force from part (b). This means that we need half the speed, or $v_0/2$. We can use conservation of energy to find the height at which the water has half its initial speed:

$$\frac{\mathrm{d}m \cdot v_0^2}{2} = \mathrm{d}m \cdot gh + \frac{\mathrm{d}m(v_0/2)^2}{2}.$$

We can rearrange this to find

$$\boxed{h = \frac{3v_0^2}{8g}}.$$

(d) The marbles have a change of momentum that is doubled, because they bounce downwards with the same speed as they hit the board! Thus $\mathrm{d}p = 2\mathrm{d}m \cdot v_0$ and we can apply the same logic as in part (b) to obtain

$$\boxed{m = \frac{2v_0 R}{g}}.$$

Question 3 **20pts**

Find the center of mass of the following objects:

(a) A cuboid of uniform mass density, ρ, with height H and a square base of side length a. For your calculation, assume that the cuboid is oriented such that the height is defined to be in the z direction and that the origin is defined at one corner of the cuboid.

(b) A cuboid of nonuniform mass density, $\rho(x, y, z) = mz^2$, with height H and a square base of side length a. Here m is a constant. For your calculation, assume that the cuboid is oriented such that the height is defined to be in the z direction and that the origin is defined at one corner of the cuboid.

(c) A pyramid of nonuniform mass density, $\rho(x, y, z) = mz^2$, with a square base of side length a and four triangular faces that are equilateral triangles. Here m is a constant. For your calculation, assume that the pyramid is oriented such that the height is defined to be in the z direction and the origin lies directly below the point of the pyramid.

Solution 3

For the following solutions, we always start from the definition of the centre of mass

$$\vec{r}_{\mathrm{CM}} = \frac{1}{M}\int \vec{r}\mathrm{d}m = \frac{1}{M}\int \vec{r}\cdot\rho(\vec{r})\mathrm{d}V.$$

(a) For our first example, the density is constant, so we can figure out the solution pretty quickly by symmetry arguments. However, I will do it the long way, just to illustrate the method. We start with

$$\vec{r}_{\mathrm{CM}} = \frac{\rho}{M}\int \vec{r}\mathrm{d}V.$$

For a cuboid, it makes sense to use Cartesian coordinates, so

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

and the coordinate of the infinitesimal volume is

$$\vec{r} = (x, y, z).$$

Taking each component of the centre of mass in turn, we have

$$\begin{aligned} x_{\text{CM}} &= \frac{\rho}{M}\int_0^H\int_0^a\int_0^a x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \frac{\rho}{M}\int_0^H \mathrm{d}z\int_0^a \mathrm{d}y\int_0^a x\,\mathrm{d}x \\ &= \frac{\rho}{M}H\cdot a\cdot\left.\frac{x^2}{2}\right|_0^a \\ &= \frac{\rho H a^3}{2M}. \end{aligned}$$

We can simplify this by noting that

$$\frac{\rho}{M} = \frac{1}{V} = \frac{1}{Ha^2},$$

so

$$x_{\text{CM}} = \frac{Ha^3}{2}\frac{1}{Ha^2} = \frac{a}{2}.$$

This is as expected! By symmetry, we must also have

$$y_{\text{CM}} = \frac{a}{2}.$$

Finally,

$$\begin{aligned} z_{\text{CM}} &= \frac{\rho}{M}\int_0^H\int_0^a\int_0^a z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \frac{\rho}{M}\int_0^H z\,\mathrm{d}z\int_0^a \mathrm{d}y\int_0^a \mathrm{d}x \\ &= \frac{\rho}{M}\cdot\left.\frac{z^2}{2}\right|_0^H\cdot a\cdot a \\ &= \frac{\rho H^2 a^2}{2M} \\ &= \frac{H}{2}. \end{aligned}$$

Putting this all together, we obtain an answer that could have been more easily found using symmetry:

$$\boxed{\vec{r}_{\text{CM}} = \frac{1}{2}\Big(a, a, H\Big)}.$$

(b) For our second example, the density is no longer constant, but it does not depend on x or y, so we can carry out those integrals as before. However, we do have to adjust our z integrals!

We find

$$
\begin{aligned}
x_{\mathrm{CM}} &= \frac{1}{M}\int_0^H \int_0^a \int_0^a x \, \mathrm{d}x \, \mathrm{d}y \, m z^2 \mathrm{d}z \\
&= \frac{m}{M}\int_0^H z^2 \, \mathrm{d}z \int_0^a \mathrm{d}y \int_0^a x \, \mathrm{d}x \\
&= \frac{m}{M} \cdot \frac{z^3}{3}\bigg|_0^H \cdot a \cdot \frac{x^2}{2}\bigg|_0^a \\
&= \frac{mH^3a^3}{6M}.
\end{aligned}
$$

By symmetry, this means

$$
y_{\mathrm{CM}} = \frac{mH^3a^3}{6M},
$$

as well.

The third coordinate of the centre of mass is given by

$$
\begin{aligned}
z_{\mathrm{CM}} &= \frac{m}{M}\int_0^H \int_0^a \int_0^a z^3 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \\
&= \frac{m}{M}\int_0^H z^3 \, \mathrm{d}z \int_0^a \mathrm{d}y \int_0^a \mathrm{d}x \\
&= \frac{m}{M} \cdot \frac{z^4}{4}\bigg|_0^H \cdot a \cdot a \\
&= \frac{mH^4a^2}{4M}.
\end{aligned}
$$

Putting this all together, we obtain an answer that could have been more easily found using symmetry:

$$
\boxed{\vec{r}_{\mathrm{CM}} = \frac{mH^3a^2}{M}\left(\frac{a}{6}, \frac{a}{6}, \frac{H}{4}\right)}.
$$

(c) First, we should recognise that the x and y coordinates of the centre of mass will lie at $(x, y) = (0, 0)$, by symmetry. So we only need to worry about the z direction.

Let's do a little trigonometry. The side length of the equilateral triangles is $\sqrt{3}a/2$. This means the height of the pyramid is

$$
h^2 + \left(\frac{a}{2}\right)^2 = \left(\frac{\sqrt{3}a}{2}\right)^2,
$$

or

$$
h = \frac{a}{\sqrt{2}}.
$$

This will be the maximum value of z for our z integral.

We also need to find the limits for the x and y integrals. We start by working at $y = 0$. In this plane, the sides of the pyramid form a triangle that has equation

$$z = h - \frac{2h}{a}x,$$

for $x > 0$, and

$$z = h + \frac{2h}{a}x,$$

for $x < 0$.

Solving these for x as a function of z gives

$$x = \frac{a}{2} - \frac{az}{2h} = \frac{a}{2} - \frac{z}{\sqrt{2}}$$

and

$$x = \frac{az}{2h} - \frac{a}{2} = \frac{z}{\sqrt{2}} - \frac{a}{2},$$

respectively.

By symmetry, we must have the same equations in the y direction, too:

$$\begin{aligned} y &= \frac{a}{2} - \frac{z}{\sqrt{2}} \qquad && y > 0, \\ y &= \frac{z}{\sqrt{2}} - \frac{a}{2} \qquad && y < 0. \end{aligned}$$

Therefore our integral becomes

$$\begin{aligned} z_{\text{CM}} &= \frac{m}{M} \int_0^{a/\sqrt{2}} \int_{z/\sqrt{2}-a/2}^{a/2-z/\sqrt{2}} \int_{z/\sqrt{2}-a/2}^{a/2-z/\sqrt{2}} z^3 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \\ &= \frac{m}{M} \int_0^{a/\sqrt{2}} \int_{z/\sqrt{2}-a/2}^{a/2-z/\sqrt{2}} z^3 \left(a - \sqrt{2}z\right) \, \mathrm{d}y \, \mathrm{d}z \\ &= \frac{m}{M} \int_0^{a/\sqrt{2}} z^3 \left(a - \sqrt{2}z\right)^2 \, \mathrm{d}z \\ &= \frac{m}{M} \left(\frac{a^2 z^4}{4} - \frac{2\sqrt{2}az^5}{5} + \frac{z^6}{3} \right) \Bigg|_0^{a/\sqrt{2}} \\ &= \frac{m}{M} \cdot \frac{a^6}{240}. \end{aligned}$$

We can now write down our final result. The centre of mass is at

$$\boxed{\vec{r}_{\text{CM}} = \left(0, 0, \frac{ma^6}{240M}\right)}.$$