Physics 101H

General Physics 1 – Honors

Lecture 44 – 11/20/23

Oscillations

Summary

Topics

Friday: Oscillations [chapter 15]

- SHM and uniform circular motion

Today: Oscillations [chapter 16]

- Damped oscillations
- Forced oscillations

Check your understanding

Instructions: Pause the lecture and solve the following question. Your answers will not be graded; your discussion is for your own learning. It is ok if you do not complete the question, but make sure you identify the key steps and write down the main equations.

Problem 44.1: The expression for the angle of an oscillating pendulum with small amplitude is $\theta(t) = \theta_0 \cos(\omega t+\varphi)$. Is the angular velocity $d\theta/dt$ equal to the angular velocity ω?

(a) Yes

(b) No

(c) I'm sorry, what?

Check your understanding

Instructions: Pause the lecture and solve the following question. Your answers will not be graded; your discussion is for your own learning. It is ok if you do not complete the question, but make sure you identify the key steps and write down the main equations.

Problem 44.2: The position of a particle is given by $x(t) = x_0\cos(\omega t + \pi/6)$, where $x_0 = 6$ m and $\omega = 2$ rad/s. Find:

(a) the position of the particle at 10s;
(b) the speed of the particle at 10s;
(c) its maximum speed (at any possible time)?

Check your understanding

Instructions: Pause the lecture and solve the following question. Your answers will not be graded; your discussion is for your own learning. It is ok if you do not complete the question, but make sure you identify the key steps and write down the main equations.

Problem 44.3:

(a) If you plot the functions $\cos(\omega t)$ and $\cos(\omega t + \pi/4)$, the second curve ($\cos(\omega t + \pi/4)$) will be shifted relative to the first curve ($\cos(\omega t)$). But which way?

(i) left

(ii) right

(b) If two harmonic oscillators have positions (as a function of time) given by $x_1(t) = \cos(\omega t)$ and $x_2(t) = \cos(\omega t + \pi/4)$, which has the greatest maximum velocity?

(i) $x_1(t)$

(ii) $x_2(t)$

(iii) neither

Check your understanding

Instructions: Pause the lecture and solve the following question. Your answers will not be graded; your discussion is for your own learning. It is ok if you do not complete the question, but make sure you identify the key steps and write down the main equations.

Problem 44.4:

Show that the energy of a simple harmonic oscillator is conserved.

Damped and forced oscillations

We happens if we add in friction or other resistive forces?

- **damped oscillations**

We can also try applying an external oscillating force (like pushing a swing)

- **forced oscillations**.

Damped and forced oscillations

Summary

Topics

Today: Oscillations [chapter 16]

- Damped oscillations
- Forced oscillations

"Wednesday": Waves [chapter 16]

- Types of waves
- Wave equation
- Solutions to the wave equation
- Wave energy

PHYSICS 101 – HONORS

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Oscillating pendulum 1 (slide 3)

No! They are not the same. Let's check:

$$\theta(t) = \theta_0 \cos(\omega t + \phi) \Rightarrow \frac{d\theta}{dt} = -\theta_0 \omega \sin(\omega t + \phi)$$

This is clearly not equal to ω, which is a constant.

$\dot{\theta}(t)$ – the rate of change of the angle with respect to vertical

ω – rate of change of $(\omega t + \phi)$, the argument of the cosine
N.B. this increases without bound for all time!

Oscillating pendulum 2 (slide 4)

(a) $x(t) = x_0 \cos(\omega t + \pi/6)$

$\Rightarrow x(t = 10s) = 6 \cdot \cos(2 \cdot 10 + \pi/6) = \underline{-0.62\ m}$

(b) $v(t) = \frac{dx}{dt} = -x_0 \omega \sin(\omega t + \pi/6)$

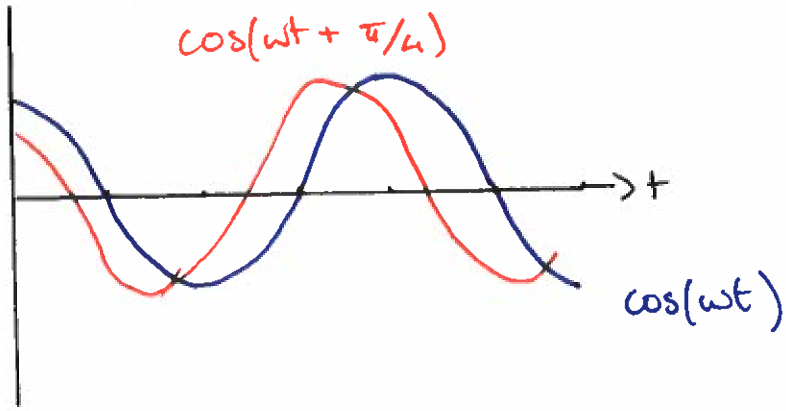
$\Rightarrow v(t = 10s) = -6 \cdot 2 \sin(2 \cdot 10 + \pi/6) = \underline{11.9\ m/s}$

(c) Maximum value of $v(t)$ occurs when $\sin(\omega t + \pi/6) = 1$

$\Rightarrow v_{max} = -x_0 \omega = \underline{-12\ m/s}$

Harmonic oscillator (slide 5)

(a)



Answer is <u>(i)</u>, because at $t = 0$ we have

$$\cos(\omega t) = 1$$
$$\cos(\omega t + \pi/4) = \cos(\pi/4) = 0.707\ldots$$

Note $\cos(\omega t + \pi/4) = 1$ at $\omega t = -\pi/4$ or $t = -\frac{\pi}{4\omega}$

to the left of $t = 0$.

(b) $x_1(t) = \cos(\omega t) \Rightarrow v_1(t) = \dot{x}_1(t) = -\omega \sin(\omega t)$

$x_2(t) = \cos(\omega t + \pi/4) \Rightarrow v_2(t) = \dot{x}_2(t) = -\omega \sin(\omega t + \pi/4)$

Both have the <u>same</u> $|v_{max}| = \omega$, so the answer is <u>(iii)</u>, but note that they reach their maximum speed at different times.

Energy example (slide 6)

Mechanical energy is

$$E_M = E_K + E_p$$

$$= \frac{1}{2} m v^2(t) + \frac{1}{2} k x^2(t)$$

$$= \frac{1}{2} m \left[-A\omega \sin(\omega t + \phi)\right]^2 + \frac{1}{2} k \left[A \cos(\omega t + \phi)\right]^2$$

$$= \frac{1}{2}\left[m \; A^2 \omega^2 \sin^2(\omega t + \phi) + k A^2 \cos^2(\omega t + \phi)\right]$$

$$= \frac{1}{2}\left[m A^2 \frac{k}{m} \sin^2(\omega t + \phi) + k A^2 \cos^2(\omega t + \phi)\right]$$

$$= \frac{1}{2}\left[k A^2 \sin^2(\omega t + \phi) + k A^2 \cos^2(\omega t + \phi)\right]$$

$$= \frac{1}{2} k A^2 \underbrace{\left[\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)\right]}_{\sin^2(x) + \cos^2(x) = 1}$$

$$= \frac{1}{2} k A^2$$

constant!

Damped oscillations (slide 5)

So far we have considered a simple oscillator without any friction or resistive force. So let's add in drag (either air resistance, or a spring submerged in fluid)

Now the net force is

$F_{net} = -kx - bv$ $\Rightarrow$ Newton's 2^{nd} law means

$$-kx - bv = ma$$

$$\Rightarrow -kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

We write this as

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

$$\frac{dx}{dt} = \alpha e^{\alpha t}$$
$$\frac{d^2x}{dt^2} = \alpha^2 e^{\alpha t}$$

Our Ansatz is $x(t) = e^{\alpha t}$

"guess at a solution"

$$\Rightarrow m\alpha^2 e^{\alpha t} + b\alpha e^{\alpha t} + k e^{\alpha t} = 0$$

$$\Rightarrow (m\alpha^2 + b\alpha + k)e^{\alpha t} = 0$$

For this to hold for all times this means

$$m\alpha^2 + b\alpha + k = 0$$

$$\alpha = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} = -\frac{b}{2m} \pm \sqrt{\frac{b^2 - 4mk}{4m^2}}$$

Let's define $\omega^2 \equiv \frac{b^2}{4m^2} - \frac{k}{m}$

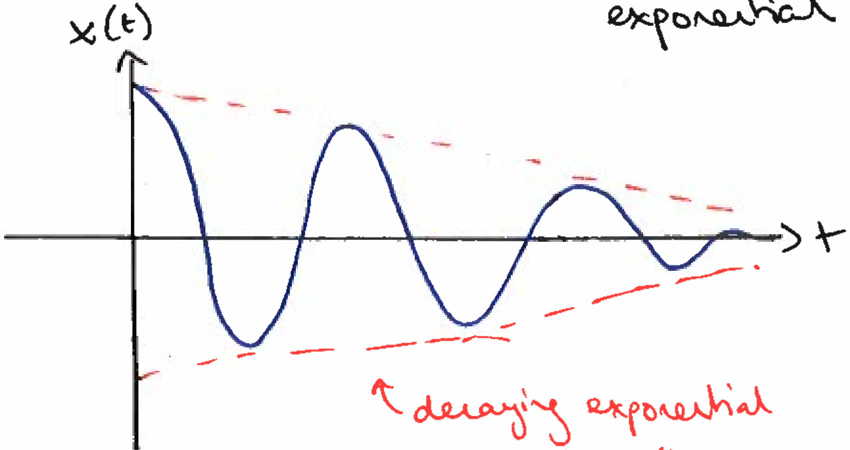
Then $x(t) = e^{-\frac{bt}{2m} \pm\sqrt{\omega^2}\,t}$

If the damping is relatively small then $k \gg b$ and

$\omega^2 < 0 \Rightarrow \sqrt{\omega^2} = \pm i\omega$

and

$$x(t) = e^{-\frac{b}{2m}t \pm i\omega t} = \underbrace{e^{-\frac{b}{2m}t}}_{\text{decaying exponential}} \underbrace{e^{\pm i\omega t}}_{\text{oscillating term}}$$



$$\omega = \sqrt{\underbrace{\frac{k}{m}}_{\substack{\omega_0^2 \\ \text{"natural} \\ \text{frequency"}}} - \underbrace{\left(\frac{b}{2m}\right)^2}_{\substack{\text{damping} \\ \text{term}}}}$$

$\omega_0 > b/2m$ = "underdamped"

$\omega_0 < b/2m$ = "overdamped"

$\omega_0 = b/2m$ = "critically damped" (no oscillations)

Most general solution is

$$x(t) = \underbrace{Ae^{-bt/2m}}_{\text{damping term}} \underbrace{\cos(\omega t + \phi)}_{\text{oscillating term}}$$

Forced Oscillations (slide 5)

We can drive an oscillator with an external force

$F_D = F \sin \omega t$ ← imagine periodically pushing a swing

=> net force is sum of driving force, damping force
and oscillatory (spring) force

$F_{net} = F \sin \omega t - bv - kx$

Applying Newton's second law

=> $F \sin \omega t - b \cdot v - kx = Ma$

=> $F \sin \omega t - b\dot{x} - kx = M\ddot{x}$

or $\ddot{x} + \frac{b}{M}\dot{x} + \frac{k}{M}x = F \sin \omega t$ recall that $\ddot{x} - \omega^2 x = 0$ is a homogeneous ODE

The solution to this inhomogeneous ordinary differential equation turns out to be

$$x(t) = \frac{F}{M\sqrt{(\omega - \omega_0)^2 + b^2\omega^2/M^2}} \cdot \cos(\omega t + \phi)$$

This is strongly peaked around $\omega \simeq \omega_0$ - a phenomenon called resonance. We typically refer to ω_0 as the resonant frequency or natural frequency. And ω is the driving frequency.