Physics 101H

General Physics 1 – Honors

Lecture 42 – 11/16/23

Simple Harmonic Motion

Summary

Topics

Yesterday: Fluid dynamics [chapter 14]

- Continuity equation
- Bernoulli equation

Today: Oscillations [chapter 15]

- Mass on a spring
- Simple harmonic motion

Announcements

Today:	**Quiz 8**
Tomorrow:	**No office hours**
Next week:	**All classes remote**

Example 42.1: A farm maintains a large tank with an open top containing water. The water can drain through a hose of diameter 6.6 cm. The hose ends with a nozzle of diameter 2.2 cm. A rubber stopper is inserted into the nozzle and the water level is 7.5 m above the nozzle.

(a) Calculate the frictional force exerted on the stopper by the nozzle.

(b) If the stopper is removed, what mass of water flows out in two hours?

Mass on a spring

Oscillations are **periodic** motions of a system (they repeat with constant **period**)

- For example, going around a circle at a constant speed

Everything that we need to know about oscillations is actually captured by the motion of a mass on a spring – mathematically they are entirely equivalent

The more general name for this behaviour is **simple harmonic motion**

Quiz 8

You got this!

Put your name on the sheet.

Do not turn over the sheet until I tell you.

You have ten minutes to answer three questions.

You may not use notes, slides, textbook or any other resources.

Calculators are allowed, but no phones, laptops, tablets or other devices.

Summary

Topics

Today: Oscillations [chapter 15]

- Mass on a spring
- Simple harmonic motion

Tomorrow: Oscillations [chapter 15]

- SHM and uniform circular motion

Announcements

Tomorrow:	**No office hours**
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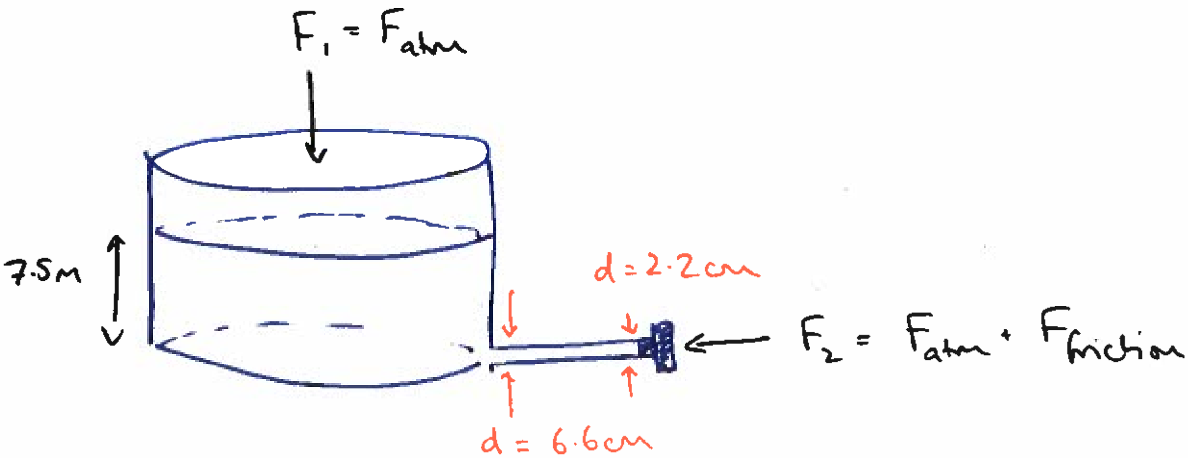
PHYSICS 101 - HONORS

Lecture 42 11/16/23

Water Tank Example (slide 3)



Apply Bernoulli's equation $P_1 + \frac{\rho v_1^2}{2} + \rho g y_1 = P_2 + \frac{\rho v_2^2}{2} + \rho g y_2$

With stopper : $v_1 = v_2 = 0$ $y_2 = 0$
$y_1 = 7.5m$ $P_2 = P_{atm} + P_{friction}$
$P_1 = P_{atm}$

$$\Rightarrow P_{atm} + \rho g y_1 = P_{atm} + P_{friction}$$

$$\Rightarrow P_{friction} = \rho g y_1$$

But $P_{friction} = \frac{F_{friction}}{A_{stopper}} \quad (= \rho g y_1)$

$$\Rightarrow F_{friction} = \rho g y_1 A_{stopper}$$
$$= 10000 \cdot 9.81 \cdot 7.5 \cdot \left(\pi \cdot \left(\frac{d}{2}\right)^2\right)$$
$$= 73575 \cdot \pi \cdot 0.011^2$$
$$= \underline{28.0 \text{ N}}$$

Once the stopper is removed we have

$P_2 = P_{atm}$ and $v_2 \neq 0$

We assume that we can approximate v_1 as 0, because the tank is so enormous (recall $v_1 = \frac{A_2 v_2}{A_1}$)

$$\Rightarrow \quad P_1 + \tfrac{1}{2}\rho \underbrace{v_1^2}_{=0} + \rho g y_1 = P_2 + \tfrac{1}{2}\rho v_2^2 + \rho g \underbrace{y_2}_{=0}$$

$$\Rightarrow \quad P_{atm} + \rho g y_1 = P_{atm} + \tfrac{1}{2}\rho v_2^2$$

$$\Rightarrow \quad \tfrac{1}{2}\rho v_2^2 = \rho g y_1$$

$$\text{or} \quad v_2^2 = 2 g y_1$$

$$\Rightarrow \quad v_2 = \sqrt{2 g y_1}$$

In a time Δt, the water moves a distance $\Delta x_2 = v_2 \Delta t$

This corresponds to a volume of water $= A_2 \Delta x_2$

$$\Rightarrow \quad V = A_2 v_2 \Delta t$$

This volume has mass $= \rho V$ or

$$M = \rho A_2 v_2 \Delta t$$

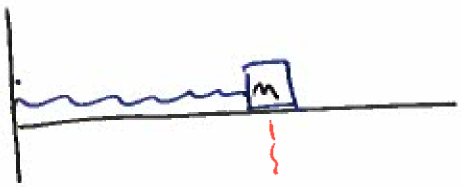
$$= 1000 \cdot \underbrace{(\pi \cdot 0.011^2)}_{=A_2} \cdot \underbrace{\sqrt{2 \cdot 9.81 \cdot 7.5}}_{=v_2} \cdot (2 \cdot 60 \cdot 60)$$

2 hours, 1 hour = 60 mins, 1 min = 60s

$$= \underline{\underline{33200 \text{ kg}}}$$

Mass on a spring (slide 4)

Let's look at a spring on a frictionless table



Only force is $\bar{F}_s = -k\bar{x}$

In 1D, Newton's 2nd law gives

$F_{net} = ma \Rightarrow -kx = ma$

Since $a = \frac{d^2x}{dt^2} = \ddot{x}$

$\Rightarrow \ddot{x} = -\frac{k}{m}x \quad \leftarrow \left(\frac{d^2x}{dt^2} = -\frac{k}{m}x\right)$

We define a new quantity $\omega^2 = k/m$

$\Rightarrow \boxed{\ddot{x} = -\omega^2 x} \quad \leftarrow$ a second-order ordinary differential equation

two time derivatives

This is the equation for simple harmonic motion

or oscillation

To solve this, let's guess

$$x(t) = e^{\alpha t}$$

$$\Rightarrow \dot{x} = \frac{dx}{dt} = \alpha e^{\alpha t}$$

$$\Rightarrow \ddot{x} = \frac{d^2x}{dt^2} = \alpha^2 e^{\alpha t}$$

Plugging this into our ODE gives

$$\underbrace{\alpha^2 e^{\alpha t}}_{\ddot{x}} = -\omega^2 \underbrace{e^{\alpha t}}_{x}$$

$$\Rightarrow -\omega^2 = \alpha^2 \quad \text{or} \quad \omega^2 = -\alpha^2$$

$$\Rightarrow \omega = \pm i\alpha \qquad i^2 = -1$$

So $x(t) = e^{\pm i\omega t}$ satisfies $\ddot{x} = -\omega^2 x$!

But what does it mean to have a complex exponential?
We can understand this through the Taylor series

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4}x^4 + \dots$$

$$\Rightarrow e^{i\omega t} = 1 + i\omega t + \frac{1}{2!}(i\omega t)^2 + \frac{1}{3!}(i\omega t)^3 + \frac{1}{4!}(i\omega t)^4 + \dots$$

$$= 1 + i\omega t - \frac{1}{2!}\omega^2 t^2 - \frac{i}{3!}\omega^3 t^3 + \frac{1}{4!}\omega^4 t^4$$

$$\left.\begin{aligned} &= 1 - \frac{1}{2!}\omega^2 t^2 + \frac{1}{4!}\omega^4 t^4 + \dots \\ &\quad + i\omega t - \frac{i}{3!}\omega^3 t^3 + \dots \end{aligned}\right\} \text{separate odd and even powers of } (\omega t)$$

$$= 1 - \frac{1}{2!}\omega^2 t^2 + \frac{1}{4!}\omega^4 t^4 + \dots + i\left(\omega t - \frac{1}{3!}\omega^3 t^3 + \dots\right)$$

But remember that

$$\sin(x) = x - \frac{x^3}{3!} + \dots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

$$\Rightarrow \quad e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$

Euler's formula

Taking $\omega t = \pi$ in Euler's formula gives Euler's identity :

$$e^{i\pi} + 1 = 0$$

considered to be one of (if not the) most famous and beautiful in mathematics

Repeating this for $e^{-i\omega t}$ gives

$$e^{-i\omega t} = \cos(\omega t) - i\sin(\omega t)$$

We can rearrange these to be

$$\cos(\omega t) = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$$

$$\sin(\omega t) = \frac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)$$

Complex exponentials are equivalent to cosines and sines !!

Let's try solutions $x(t) = \cos(\omega t)$ and $x(t) = \sin(\omega t)$

$\Rightarrow \ddot{x} = -\omega^2 \cos(\omega t)$

$= -\omega^2 x$ ✓

$x = \cos(\omega t)$

$\dot{x} = -\omega \sin(\omega t)$

$\ddot{x} = -\omega^2 \cos(\omega t)$

$\ddot{x} = -\omega^2 \sin(\omega t)$

$= -\omega^2 x$ ✓

$x = \sin(\omega t)$

$\dot{x} = \omega \cos(\omega t)$

$\ddot{x} = -\omega^2 \sin(\omega t)$

Most general solution is $x(t) = A\cos(\omega t + \phi)$

check it for yourself !