Physics 101H

General Physics 1 – Honors

Lecture 24 – 10/11/23

Centre of mass

Summary

Topics

Monday: Collisions [chapter 9]

- Collisions in 2D

Today: Centre of mass [chapter 9]

- Centre of mass
- Calculating the centre of mass
- Centre of mass examples
- 2D collision examples

Announcements

Today: **Problem Set 4 due**

No problem set assigned!

Tomorrow: **Fall Break!**

Centre of mass

Centre of mass is a special point in a system

If all the mass of a system is at the centre of mass, then the translational motion of the system is unchanged

- System moves as if any net force were applied to a particle of total mass of the system located at that point
- Centre of mass is approximately the "average position" of the system's mass

Calculating the centre of mass

Consider:

- system of point-like discrete particles
- continuous distribution of mass – an **extended object**

Remember – the system moves as though any net force were applied to a single point-like particle of equivalent mass at the centre of mass

Example 24.1: Find the centre of mass of a rod of length l and mass M.

Example 24.2: A projectile of mass 5 kg explodes into two fragments at some point in its trajectory. One fragment of mass 2 kg falls at a point $2R/3$, where R is the range of the projectile. Where does the other fragment fall, assuming no mass is lost in the process?

Example 24.3: A mass m collides elastically with a stationary mass $2m$. Assuming that the final energies turn out to be equal, find the two final speeds and angles of deflection.

Summary

Topics

Today: Centre of mass [chapter 9]

- 2D collision examples
- Centre of mass examples

Monday: Rockets [chapter 9]

- Rockets
- Propulsion and conservation of momentum
- One more centre of mass example

Announcements

Today: **Problem Set 4 due**
No problem set assigned!

Tomorrow: **Fall Break!**

Happy Fall Break!

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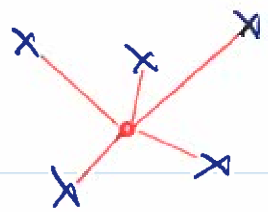
PHYSICS 101 - HONORS

Lecture 24 10/11/23

Centre of mass (slide 3)



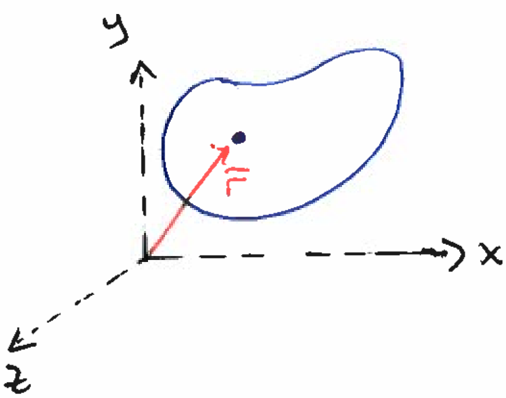
Coordinates of the centre of mass

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + \dots + m_n x_n}{m_1 + m_2 + \dots + m_n}$$

$$= \sum_{i=1}^{n} \frac{m_i x_i}{M} \qquad M = \sum_{i=1}^{n} m_i$$

$$y_{cm} = \sum_{i=1}^{n} \frac{m_i y_i}{M}$$

$$z_{cm} = \sum_{i=1}^{n} \frac{m_i z_i}{M}$$



Consider an infinitesimal mass at a point at position $\bar{r}$

$$\sum_i m_i x_i \longrightarrow \int dm_i \, \bar{r}_i$$

$$\Rightarrow \bar{r}_{cm} = \frac{1}{M} \int \bar{r} \, dm$$

Centre of mass motion given by

$$\bar{v}_{cm} = \frac{d\bar{r}_{cm}}{dt} \qquad \bar{a}_{cm} = \frac{d^2 \bar{r}_{cm}}{dt^2}$$

$$\bar{v}_{cm} = \frac{1}{M} \sum_{i=1}^{n} \frac{d\bar{r}_i}{dt} = \frac{1}{M} \sum_{i=1}^{n} m_i \bar{v}_i = \frac{1}{M} \sum_{i=1}^{n} \bar{p}_i = \frac{1}{M} \bar{P}_{total}$$

The "total momentum" is the "momentum of the centre of mass motion"

$$\bar{P}_{tot} = M\bar{v}_{cm} \quad (= \bar{P}_{cm})$$

This is the momentum of a particle of mass M, following the centre of mass motion (trajectory)

Conservation of momentum tells us that

$$\bar{P}_{cm,i} = \bar{P}_{cm,f}$$

Note that the centre of mass obeys

$$\bar{F}_{net} = \frac{d}{dt}\bar{P}_{cm} = M\bar{a}_{cm}$$

Rod example (slide 5)

Let's orient our rod along the x-axis

This is a continuous distribution of matter/mass, so

$$\bar{r}_{cm} = \frac{1}{M}\int \bar{r}\, dm$$

In this case, the rod has linear mass density $\lambda = \frac{M}{\ell}$ ⇒ a small volume has mass $dm = \lambda\, dx = \frac{M}{\ell} dx$

The position of the volume dm is just x

$$\Rightarrow r_{cm} = \frac{1}{M}\int x\, dm$$

$$= \frac{1}{M}\int x \frac{M}{\ell} dx$$

$$= \frac{1}{M}\cdot\frac{M}{\ell}\int_0^\ell x\, dx$$

$$= \frac{1}{\ell}\cdot \left.\frac{x^2}{2}\right|_0^\ell$$

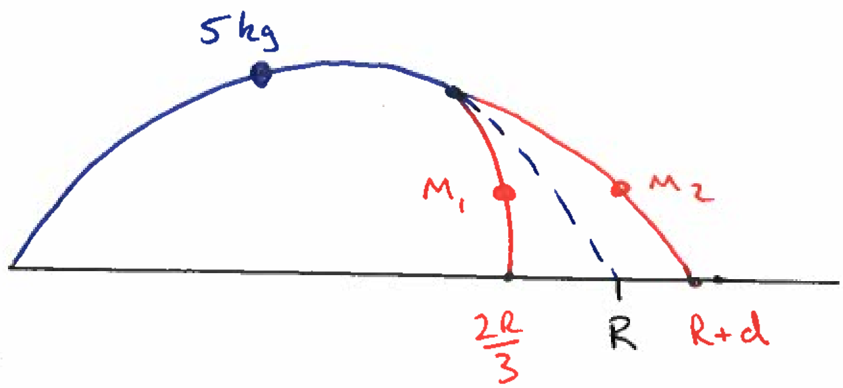
$$= \frac{1}{\ell}\cdot\frac{\ell^2}{2} = \boxed{\frac{\ell}{2}}$$

← halfway along the rod! (as we may have expected)

Projectile example (slide 6)

The key to solving this is to recognise the centre of mass continues its trajectory, even after the projectile has split!

This means the centre of mass must be at R when the two fragments land.



$M_1 = 2$ kg
$M_2 = 3$ kg

$$\Rightarrow \frac{1}{M}\cdot\left(M_1\frac{2R}{3} + M_2\cdot(R+d)\right) = R \qquad M = M_1 + M_2$$

$$\Rightarrow \frac{2M_1R}{3} + M_2R + dM_2 = MR$$

$$d = \frac{MR - \frac{2}{3}M_1R - M_2R}{M_2}$$

$$= \frac{(M - \frac{2}{3}M_1 - M_2)R}{M_2}$$

$$= \frac{5 - \frac{2}{3}\cdot 2 - 3}{3}R$$

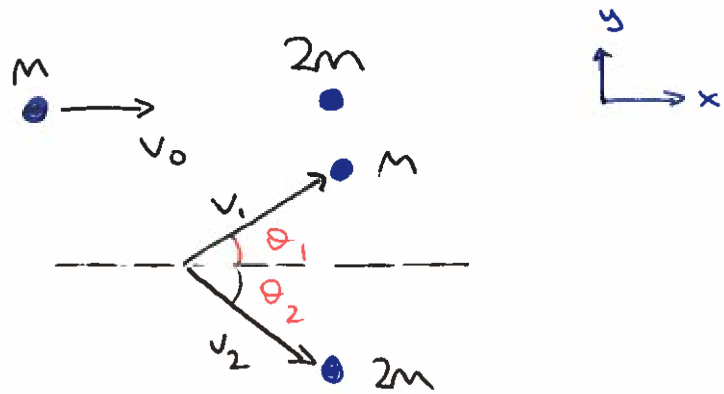
$$= \frac{2}{9}R$$

$\Rightarrow$ other projectile lands at $\boxed{\frac{11}{9}R}$

Two dimensional collision example (slide 8)

Initial :

Final :



Elastic collision ⇒ kinetic energy conserved
momentum conserved

Equal energies ⇒ $2E_K^{final} = E_K^{initial}$

$$\Rightarrow \frac{1}{2} m v_1^2 = \frac{1}{2}\left(\frac{1}{2} m v_0^2\right)$$

$$\frac{1}{2}(2m) v_2^2 = \frac{1}{2}\left(\frac{1}{2} m v_0^2\right)$$

$$\Rightarrow \boxed{v_1 = \frac{v_0}{\sqrt{2}} \qquad v_2 = \frac{v_0}{2}}$$

Conservation of momentum in $\hat{x}$: $m v_0 = m v_1 \cos\theta_1 + 2m v_2 \cos\theta_2$

$$\Rightarrow m v_0 = m \frac{v_0}{\sqrt{2}} \cos\theta_1 + 2m \frac{v_0}{2} \cos\theta_2$$

$$\Rightarrow 1 = \frac{1}{\sqrt{2}} \cos\theta_1 + \cos\theta_2$$

Conservation of momentum in $\hat{y}$: $0 = m v_1 \sin\theta_1 - 2m v_2 \sin\theta_2$

$$\Rightarrow 0 = m \frac{v_0}{\sqrt{2}} \sin\theta_1 - 2m \frac{v_0}{2} \sin\theta_2$$

$$\Rightarrow 0 = \frac{1}{\sqrt{2}} \sin\theta_1 - \sin\theta_2$$

To solve these equations we rewrite them as

$$\frac{1}{\sqrt{2}} \cos\vartheta_1 = 1 - \cos\vartheta_2$$

$$\frac{1}{\sqrt{2}} \sin\vartheta_1 = \sin\vartheta_2$$

Square them and add!

$$\frac{1}{2} \cos^2\vartheta_1 = (1 - \cos\vartheta_2)^2$$

$$\frac{1}{2} \sin^2\vartheta_1 = \sin^2\vartheta_2$$

$$\Rightarrow \frac{1}{2}\underbrace{(\cos^2\vartheta_1 + \sin^2\vartheta_1)}_{=1} = 1 - 2\cos\vartheta_2 + \underbrace{\cos^2\vartheta_2 + \sin^2\vartheta_2}_{=1}$$

$$\frac{1}{2} = 2 - 2\cos\vartheta_2$$

$$\Rightarrow \cos\vartheta_2 = \frac{3}{4} \quad \text{or} \quad \boxed{\vartheta_2 = 41.4°}$$

Plug into 1st equation

$$\frac{1}{\sqrt{2}} \cos\vartheta_1 = 1 - \cos\vartheta_2$$

$$\Rightarrow \cos\vartheta_1 = \sqrt{2} \cdot \frac{1}{4} = \frac{1}{2\sqrt{2}} \quad \text{or} \quad \boxed{\vartheta_1 = 69.3°}$$