General Physics 1–Honors (PHYS 101H): Problem Set 5–Solutions

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Overview

The written Problem Sets will help you gain experience with how to present your solutions to university-level physics problems. This will be necessary for your midterm and final exams, as well as future courses thoughout your undergraduate career. Present your solutions legibly and as logically as you can. What this means in practice is the following:

1. Write down what quantities you know.
2. Write down the relevant equations.
3. When carrying out manipulations or substituting values into equations, try to write each equality on separate lines. For example:

$$\begin{aligned} x(t_0) &= \frac{1}{2}at_0^2 + v_0t_0 + x_0 \\ &= \frac{1}{2} \cdot 9.8 \cdot 1.0^2 + 1.5 \cdot 1.0 + 0.7 \\ &= \boxed{7.1\,\mathrm{m}}. \end{aligned}$$

4. Circle or underline your final answers to identify them clearly (see the equation above)..

I will post solutions that will also provide one possible model for how to present solutions.

Some hints:

1. Only substitute values at the end of your calculation and try to carry out all manipulations symbolically.
2. Double check the order of magnitude of your answer.
3. Double check the units of your answer.
4. Double check the number of significant figures of your answer (do not give more significant figures than the question provides for physical quantities).

This Problem Set, in particular, will provide practice in applying conservation of energy to analyse motion in one and two dimensions and studying elastic collisions in one and two dimensions.

This Problem Set is worth 50 points; there are three questions in this Problem Set.

Instructions

Read these instructions carefully. You must submit your Problem Set as a **single PDF** file (it is best to use an app like Adobe Scan to make your solutions legible), with the file name `lastname_hwXX.pdf` (replace `lastname` with your last name and `XX` with the problem set number). If you do not submit your Problem Set according to these instructions, you will be deducted five points.

Question 1 **15pts**

A massless spring with spring constant k hangs vertically from a ceiling, initially at its relaxed length. A mass m is then attached to the bottom and released.

(a) Calculate the total potential energy (gravitational and spring) of the system, as a function of the height y (which is negative), relative to the initial position.

(b) Find y_0, the point at which the potential energy is at a minimum. Make a rough plot of the potential energy as a function of y. Label the plot to indicate the coordinates of the minimum and the intersection of the curve with the axes.

(c) Rewrite the potential energy as a function of $z = y - y_0$ and make a labelled rough plot of this new form of the potential, as a function of z. Explain why your result shows that a hanging spring can be considered to be considered to be a spring in a world without gravity, provided that the equilibrium point, y_0, is now called the "relaxed" length of the spring.

Solution 1

Figure 1 illustrates the spring system in question 1.

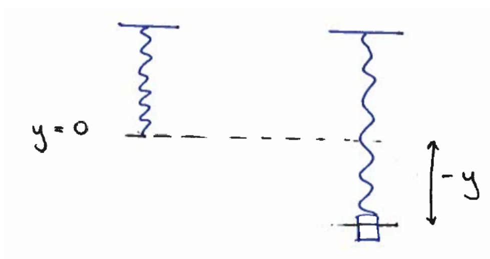

Figure 1: Diagram for Solution 1.

(a) The total potential energy is the sum of the gravitational and elastic potential energies,

$$E_P(y) = mgy + \frac{k}{2}y^2.$$

Note the first term is negative!

(b) The minimum of the potential energy is given by the condition

$$\frac{\mathrm{d}}{\mathrm{d}y}E_P(y) = 0.$$

This means that

$$mg + ky = 0.$$

The solution of this is

$$\boxed{y_0 = -\frac{mg}{k}}.$$

To draw out plot, let's calculate some values:

$$E_P(y_0) = -\frac{m^2g^2}{2k}.$$

We also know that at $y = 0$ and $y = -\frac{2mg}{k}$ we have $E_p = 0$. The plot is shown in figure 2. **Note that full marks require labelled axes as well as the coordinates of the minimum and the zero-crossings.**

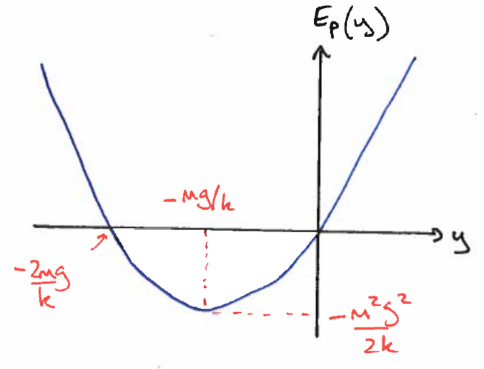

Figure 2: Plot for Solution 1(b).

(c) The new variable is $z = y - y_0$, or

$$y = z + y_0 = z - \frac{mg}{k}.$$

Plugging this into our expression for the potential, we have

$$\begin{aligned}
E_P(z) &= mg\left(z - \frac{mg}{k}\right) + \frac{k}{2}\left(z - \frac{mg}{k}\right)^2 \\
&= mgz - \frac{m^2g^2z}{k} + \frac{k}{2}\left(z^2 - \frac{2mgz}{k} + \frac{m^2g^2}{k^2}\right) \\
&= mgz - \frac{m^2g^2z}{k} + \frac{kz^2}{2} - mgz + \frac{m^2g^2}{2k} \\
&= \boxed{\frac{kz^2}{2} - \frac{m^2g^2}{2k}}.
\end{aligned}$$

The plot of this function is shown in figure 3.

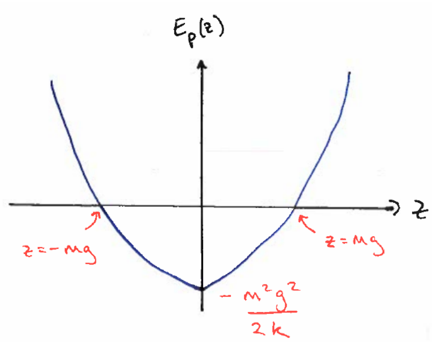

Figure 3: Plot for Solution 1(c).

The minimum of $E_p(z)$ now occurs at $z = 0$. Only *differences* in potential energies matter, so we are free to define the potential energy of the system as

$$E_p(z) = \frac{kz^2}{2}.$$

In this system, there is no mention of gravity in the expression for the potential (in other words, g does not appear at all)! At the equilibrium point, the force due to gravity on the mass is cancelled by the elastic restoring force due to the spring. The new equilibrium point of the spring (with the mass on) can be treated as the "relaxed" length and then the oscillations of the spring will be governed only by the net force, which is due to *further* stretching or compression of the spring.

Question 2 **15pts**

A bead is initially at rest at the top of a fixed frictionless hoop with radius R that lies in a vertical plane. The bead is then given an infinitesimal push so that it slides down and around the hoop.

(a) What is the speed of the bead after it has fallen through an angle θ (measured relative to the vertical).

(b) Take the time derivative of your result (don't forget to use the chain rule) to verify that the tangential acceleration dv/dt equals the tangential component of the acceleration due to gravity.

Solution 2

The geometry of the hoop is shown in figure 4.

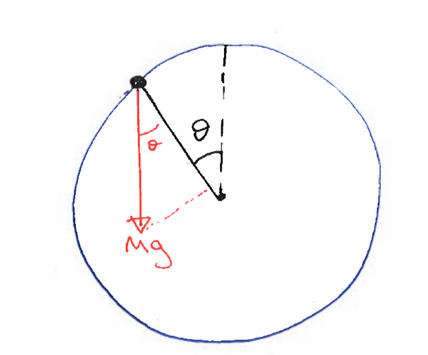

Figure 4: Plot for Solution 2.

(a) The height fallen from the top is

$$h = R - R\cos\theta = R(1 - \cos\theta).$$

We apply conservation of energy to relate the kinetic energy gained and the potential energy lost via

$$\frac{mv^2}{2} = mgh,$$

so

$$\begin{aligned} v &= \sqrt{2gh} \\ &= \boxed{\sqrt{2gR(1-\cos\theta)}}. \end{aligned}$$

(b) We need to calculate the derivative of this expression with respect to the time. The chain rule tells us that

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}\theta}\dot{\theta}.$$

The tricky part of this is

$$\frac{\mathrm{d}v}{\mathrm{d}\theta},$$

which we can calculate by writing

$$f(\theta) \equiv 1 - \cos\theta,$$

so

$$v = \sqrt{2gR} f^{1/2}.$$

We then use the chain rule to find

$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}\theta} &= \frac{\mathrm{d}v}{\mathrm{d}f}\frac{\mathrm{d}f}{\mathrm{d}\theta} \\ &= \sqrt{2gR}\frac{f^{-1/2}}{2}\sin\theta \\ &= \sqrt{\frac{gR}{2(1-\cos\theta)}}\sin\theta \end{aligned}$$

Therefore

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \sqrt{\frac{gR}{2(1-\cos\theta)}}\sin\theta\,\dot{\theta}.$$

We can simplify this by using $v = \dot{\theta}R$ to rewrite $\dot{\theta}$ as

$$\begin{aligned} \dot{\theta} &= \frac{v}{R} \\ &= \frac{\sqrt{2gR(1-\cos\theta)}}{R} \\ &= \sqrt{\frac{2g(1-\cos\theta)}{R}}. \end{aligned}$$

Plugging this into our expression for the acceleration, we have

$$\begin{aligned} a_t &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= \sqrt{\frac{gR}{2(1-\cos\theta)}}\sin\theta \cdot \sqrt{\frac{2g(1-\cos\theta)}{R}} \\ &= \boxed{g\sin\theta}. \end{aligned}$$

Looking at our diagram in 4, the acceleration due to gravity is directed vertically downwards (indicated by the weight, mg, in the diagram). Therefore the component of the weight in the tangential direction is $mg\sin\theta$. In other words, the component of the acceleration due to gravity in the tangential direction is $g\sin\theta$. Therefore the tangential acceleration a_t is equal to the tangential component of gravity, as required.

Question 3 **20pts**

In a one-dimensional collision, a mass M moving with velocity V collides with a mass m that is at rest. Assume that all collisions are elastic.

(a) Show that the resulting velocities are given by

$$V_M = \frac{(M-m)}{M+m}V, \qquad v_m = \frac{2M}{M+m}V.$$

(b) What is the relationship between these velocities if $m \ll M$? And what about if $M \ll m$?

Imagine that you now add a third stationary mass, μ, between M (still initially moving with velocity V) and m (which is still at rest).

(c) Find an expression for μ, in terms of m and M, that maximises the final velocity of m after all collisions.

Solution 3

(a) We solve this using conservation of energy and conservation of momentum, which both apply to elastic conditions. Conservation of energy tells us that

$$\frac{MV^2}{2} = \frac{MV_M^2}{2} + \frac{mv^2}{2},$$

and conservation of energy tells us

$$MV = MV_M + mv_m.$$

Let's eliminate v_m by rearranging the second of these equations as

$$v_m = \frac{M(V-V_M)}{m}.$$

We can now plug this into the conservation of energy equation to find

$$\frac{MV^2}{2} = \frac{MV_M^2}{2} + \frac{mM^2((V-V_M))^2}{2m^2},$$

which can be expanded as

$$V^2 = V_M^2 + \frac{M}{m}(V^2 - 2VV_M + V_M^2).$$

Let's collect together terms to find a quadratic equation for V_M,

$$0 = \left(\frac{M}{m}+1\right)V_M^2 - \frac{2M}{m}VV_M + \left(\frac{M}{m}-1\right)V^2,$$

or

$$0 = (M + m)\, V_M^2 - 2MVV_M + (M - m)\, V^2.$$

This can be solved by writing it as

$$0 = [(M + m)\, V_M - (M - m)\, V]\,(V_M - V).$$

The first solution, $V_M = V$ is the "trivial" solution in which no collision occurs. The other solution is

$$\boxed{V_M = \frac{(M - m)}{(M + m)} V},$$

and plugging this into our expression for v_m gives

$$\boxed{v_m = \frac{2M}{M + m} V}.$$

(b) In the limit that $m \ll M$, we have

$$\begin{aligned}
V_M &= \frac{(1 - m/M)}{(1 + m/M)} V \\
&= \left(1 - \frac{m}{M}\right)\left(1 + \frac{m}{M}\right)^{-1} V \\
&\approx \left(1 - \frac{m}{M}\right)\left(1 - \frac{m}{M} + \frac{m^2}{M^2} + \cdots\right) V \\
&= \left(1 - \frac{2m}{M} + \cdots\right) V.
\end{aligned}$$

The other velocity is

$$\begin{aligned}
v_m &= \frac{2}{1 + m/M} V \\
& 2\left(1 + \frac{m}{M}\right)^{-1} V \\
& 2\left(1 - \frac{m}{M} + \frac{m^2}{M^2} + \cdots\right) V \\
&= \left(2 - \frac{2m}{M} + \cdots\right) V.
\end{aligned}$$

If m really is tiny then the final velocities are

$$\boxed{V_M \approx V\,, \qquad v \approx 2V}.$$

In other words, the large mass continues at almost the same speed, while the smaller mass picks up twice that speed.

In the other limit, $M \ll m$, we have

$$\begin{aligned}
V_M &= \frac{(M/m - 1)}{(1 + M/m)} V \\
&= \left(\frac{M}{m} - 1\right)\left(1 + \frac{M}{m}\right)^{-1} V \\
&\approx \left(\frac{M}{m} - 1\right)\left(1 - \frac{M}{m} + \frac{M^2}{m^2} + \cdots\right) V \\
&= \left(-1 + \frac{M}{m} - \frac{M}{m} + \cdots\right) V.
\end{aligned}$$

The other velocity is

$$\begin{aligned}
v_m &= \frac{2M/m}{M/m + 1} V \\
&\frac{2M}{m}\left(1 + \frac{M}{m}\right)^{-1} V \\
&\frac{2M}{m}\left(1 - \frac{M}{m} + \frac{M^2}{m^2} + \cdots\right) V \\
&= \left(\frac{2M}{m} + \cdots\right) V.
\end{aligned}$$

If M really is tiny then the final velocities are

$$\boxed{V_M \approx -V\,, \qquad v \approx 0}.$$

In other words, the smaller mass M bounces off the very-much-larger mass and returns at the same speed, but in the opposite direction. The much larger mass hardly moves.

(c) In this new situation, we can directly apply our equations from part (a), but to a collision between M and μ. Thus the speed of the mass μ is

$$v_\mu = \frac{2M}{M + \mu} V.$$

This is now the *initial* speed of the incoming mass (μ) in the second collision. So the resulting velocity for m, after the second collision is

$$v_m = \frac{2\mu}{\mu + m} v_\mu = \frac{2\mu}{\mu + m} \frac{2M}{M + \mu} V.$$

We want to maximise this with respect to the mass μ, so we need to determine the solution of

$$\frac{\mathrm{d}}{\mathrm{d}\mu} \frac{2\mu}{\mu + m} \frac{2M}{M + \mu} V = 0.$$

The derivative is given by the quotient rule and is

$$\frac{\mathrm{d}}{\mathrm{d}\mu} \frac{2\mu}{\mu + m} \frac{2M}{M + \mu} V = 4MV \left[\frac{(\mu + m)(M + \mu) \cdot 1 - \mu(2\mu + m + M)}{(\mu + m)^2 (M + \mu)^2}\right].$$

For this to be zero, we must have that the numerator is zero. This means

$$(\mu + m)(M + \mu) - \mu(2\mu + m + M) = 0,$$

or

$$\mu^2 + \mu(m + M) + mM - 2\mu^2 - \mu(m + M) = -\mu^2 + mM = 0.$$

This has two solutions, but the negative mass solution is unphysical. Thus the mass that maximises the speed of m at the end of the two collisions is

$$\boxed{\mu = \sqrt{mM}}.$$