General Physics 1–Honors (PHYS 101H): Problem Set 2–Solutions

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Overview

The written Problem Sets will help you gain experience with how to present your solutions to university-level physics problems. This will be necessary for your midterm and final exams, as well as future courses thoughout your undergraduate career. Present your solutions legibly and as logically as you can. What this means in practice is the following:

1. Write down what quantities you know.
2. Write down the relevant equations.
3. When carrying out manipulations or substituting values into equations, try to write each equality on separate lines. For example:

$$\begin{aligned} x(t_0) &= \frac{1}{2}at_0^2 + v_0 t_0 + x_0 \\ &= \frac{1}{2} \cdot 9.8 \cdot 1.0^2 + 1.5 \cdot 1.0 + 0.7 \\ &= \boxed{7.1\,\mathrm{m}}. \end{aligned}$$

4. Circle or underline your final answers to identify them clearly (see the equation above)..

I will post solutions that will also provide one possible model for how to present solutions.

Some hints:

1. Only substitute values at the end of your calculation and try to carry out all manipulations symbolically.
2. Double check the order of magnitude of your answer.
3. Double check the units of your answer.
4. Double check the number of significant figures of your answer (do not give more significant figures than the question provides for physical quantities).

This Problem Set, in particular, will provide practice in applying the equations for kinematics to problems in two dimensions and allow you to practice analysing circular motion.

This Problem Set is worth 50 points; there are three questions in this Problem Set.

Instructions

Read these instructions carefully. You must submit your Problem Set as a **single PDF** file (it is best to use an app like Adobe Scan to make your solutions legible), with the file name `lastname_hwXX.pdf` (replace `lastname` with your last name and `XX` with the problem set number). If you do not submit your Problem Set according to these instructions, you will be deducted five points.

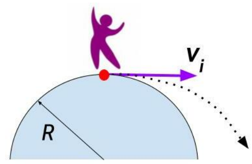

Figure 1: Person kicking a pebble on a boulder for Problem 1. I don't know why they are doing this.

Question 1 **20pts**

A person standing on top of a hemispherical boulder of radius R kicks a pebble (see Figure 1), initially at rest on top of the boulder, with an initial horizontal velocity $\overline{v}_i$.

(a) What is the minimum value of the initial speed, v_i, that ensures the pebble does not touch the boulder after it is kicked?

(b) With this initial speed (i.e. your answer to part a), where does the pebble hit the ground?

Solution 1

(a) Let's start by setting up our coordinate system (our reference frame). We define the horizontal and vertical components as x and y respectively, and assume that the motion occurs in a two-dimensional plane. For the pebble not to hit the surface of the boulder, we must have

$$x^2 + y^2 > R^2,$$

for $t > 0$.

The acceleration is given by

$$\mathbf{a} = -g\hat{y},$$

and the initial velocity is

$$\mathbf{v}_i = v_i\hat{x}.$$

Thus, the equation of motion in the x direction is

$$x = v_i t,$$

which tells us that $t = x/v_i$, and the equation of motion in the y direction is

$$y = -\frac{gt^2}{2} + R.$$

Substituting our value of t into this equation for y, we obtain

$$y = -\frac{g}{2}\left(\frac{x}{v_i}\right)^2 + R,$$

or

$$x^2 = \frac{2v_i^2}{g}(R - y).$$

Now we apply our constraint condition, so

$$x^2 + y^2 = \frac{2v_i^2}{g}(R - y) + y^2 > R^2,$$

or

$$\frac{2v_i^2}{g}(R - y) > R^2 - y^2 = (R + y)(R - y).$$

Assuming $y \neq R$, we can cancel the factor of $(R - y)$ on both sides to find

$$\frac{2v_i^2}{g} > R + y,$$

or

$$v_i^2 > \frac{g(R + y)}{2} \qquad \Rightarrow \qquad v_i > \sqrt{\frac{g(R + y)}{2}}.$$

For all $t > 0$, the maximum height is the initial height, so $y < R$. This means the maximum value that the initial velocity can take is at $y = R$. So

$$v_i > \sqrt{\frac{2gR}{2}}$$

or

$$\boxed{v_i > \sqrt{gR}}.$$

(b) With this initial speed in the y direction, we can use the kinematic equation in the vertical direction to find the distance (x) until the pebble hits the ground ($y = 0$):

$$y = 0 = -\frac{g}{2}\left(\frac{x}{v_i}\right)^2 + R.$$

We rearrange this as

$$\frac{g}{2}\left(\frac{x}{v_i}\right)^2 = R,$$

and substitute our value of $v_i = \sqrt{gR}$ to give

$$\frac{g}{2}\left(\frac{x}{\sqrt{gR}}\right)^2 = R,$$

or

$$\frac{gx^2}{2gR} = R.$$

This we solve to find

$$x = \pm\sqrt{2}R.$$

The negative solution is the solution in the opposite direction, so we take the positive solution. This means the pebble lands at

$$\boxed{x = \sqrt{2}R}.$$

An alternative acceptable answer is the distance beyond the boulder, which is

$$\boxed{x = (\sqrt{2} - 1)R}.$$

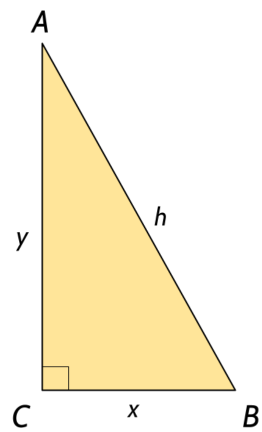

Figure 2: Triangle for Question 3.

Question 2 **15pts**

In the vertical right-angled triangle illustrated in Figure 2, a particle falls from A to B either along the hypotenuse, or along the two legs (of lengths y and x) via the third point, C. Assume that there is no friction.

(a) What is the time (call it t_H) if the particle travels along the hypotenuse?

(b) What is the time (call it t_L) if the particle travels along the legs, AC then CB? Assume that at C there is an infinitesimal curved arc that allows the change of the particle's motion to change from vertical to horizontal without any change in its speed. Neglect the time it takes to travel this infinitesimal arc.

(c) Show that $t_H = t_L$ when $x = 0$.

(d) How do t_H and t_L compare in the limit $y \ll x$?

(e) Excluding the $x = 0$ case, what triangle shape yields $t_H = t_L$?

Solution 2

(a) For this first part of the problem, it is simplest to pick a tilted coordinate frame, such that the horizontal axis is parallel to the hypotenuse. The component of the acceleration due to gravity parallel to the hypotenuse is

$$g \sin\theta = g \cdot \frac{y}{h} = g \cdot \frac{y}{\sqrt{x^2 + y^2}}.$$

The distance travelled along this hypotenuse is

$$h = \sqrt{x^2 + y^2},$$

and according to the equation of motion in this direction,

$$h = \frac{1}{2}at^2 = \frac{gy}{2\sqrt{x^2 + y^2}} t_H^2.$$

Rearranging this, we find

$$\boxed{t_H = \sqrt{\frac{2(x^2 + y^2)}{gy}}}.$$

(b) The time to fall along AC is given by

$$y = \frac{1}{2} g t_1^2,$$

which we rearrange as

$$t_1 = \sqrt{\frac{2y}{g}}.$$

At the end of this leg, the particle has final velocity $v_f = g t_1$. This is then the initial velocity for the second leg, CB. There is no acceleration along this second leg, so the particle travels a distance x with constant velocity $\sqrt{2gy}$. Thus the time taken for the second leg is

$$t_2 = \frac{x}{\sqrt{2gy}},$$

and the total time is

$$\boxed{t_L = \sqrt{\frac{2y}{g}} + \frac{x}{\sqrt{2gy}}}.$$

(c) When $x = 0$, the times are given by

$$\boxed{t_L = t_H = \sqrt{\frac{2y}{g}}},$$

as required.

(d) When $x \gg y$, the two times reduce to

$$t_L \simeq \frac{x}{\sqrt{2gy}},$$

and

$$t_H \simeq x\sqrt{\frac{2}{gy}}.$$

Therefore, in this limit,

$$\boxed{t_H \simeq 2t_L}.$$

We can interpret this as follows. For t_L, the particle moves at the maximum speed of $\sqrt{2gy}$ for almost the entire time. For t_H, the particle reaches the same maximum speed, and the average speed is half the maximum speed (true because the acceleration is constant).

(e) If we set $t_H = t_L$, then we can derive the condition for this equality to occur. We have

$$\sqrt{\frac{2(x^2+y^2)}{gy}} = \sqrt{\frac{2y}{g}} + \frac{x}{\sqrt{2gy}},$$

which we can rewrite as

$$2\sqrt{\frac{(x^2+y^2)}{2gy}} = \frac{2y}{\sqrt{2gy}} + \frac{x}{\sqrt{2gy}}.$$

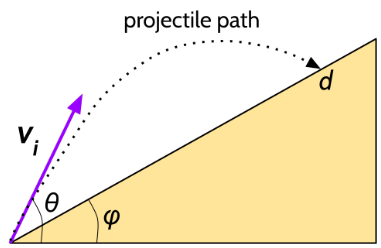

Figure 3: Triangle for Question 3.

This means

$$2\sqrt{x^2+y^2} = 2y + x,$$

or

$$4(x^2+y^2) = (2y+x)^2.$$

Multiplying out the right-hand side and rearranging, we find

$$3x^2 = 4xy,$$

or

$$\boxed{x = \frac{4y}{3}}.$$

Question 3 **15pts**

A projectile is fired up an incline of angle ϕ., with an initial velocity $\mathbf{v}_i$ at an angle θ to the horizontal. Note that $\theta > \phi$.

(a) Show that the projectile travels a distance d up the incline, where d is given by

$$d = \frac{2v_i^2 \cos\theta \sin(\theta - \phi)}{g\cos^2\phi}.$$

(b) For what value of θ is d a maximum, and what is that maximum value?

Solution 3

(a) For this solution, we will choose a reference frame in which the x axis lies along the horizontal direction and the y direction is vertical.

The coordinates of the point at which the projectile hits the incline plane are

$$(x, y) = (d\cos\phi, d\sin\phi).$$

The initial velocity and acceleration are

$$\mathbf{v}_i = v_i\cos\theta\hat{x} + v_i\sin\theta\hat{y}, \qquad \mathbf{a} = -g\hat{y}.$$

Thus the kinematic equations in the x and y directions are, respectively,

$$x = v_i(\cos\theta)t$$
$$y = -\frac{gt^2}{2} + (v_i\sin\theta)t + 0.$$

We can rearrange the first of these and then plug it into the second to obtain

$$y = -\frac{g}{2}\left(\frac{x}{v_i \cos\theta}\right)^2 + (v_i \sin\theta)\left(\frac{x}{v_i \cos\theta}\right).$$

We want to find an expression for d, so let's substitute our expression for the coordinates of the point at which the projectile hits the plane:

$$d \sin\phi = -\frac{g}{2}\left(\frac{d\cos\phi}{v_i \cos\theta}\right)^2 + (v_i \sin\theta)\left(\frac{d\cos\phi}{v_i \cos\theta}\right).$$

We can rearrange this and simplify the fractions to obtain

$$\begin{aligned} 0 &= -\frac{g}{2}\frac{d^2\cos^2\phi}{v_i^2\cos^2\theta} + d(\cos\phi\tan\theta - \sin\phi) \\ &= d\left(\frac{gd\cos^2\phi}{2v_i^2\cos^2\theta} - \cos\phi\tan\theta + \sin\phi\right). \end{aligned}$$

This means either $d = 0$ or

$$\frac{gd\cos^2\phi}{2v_i^2\cos^2\theta} - \cos\phi\tan\theta + \sin\phi = 0.$$

We are interested in the second solution, so let's rearrange this as

$$\frac{gd\cos^2\phi}{2v_i^2\cos^2\theta} = \cos\phi\tan\theta - \sin\phi,$$

or

$$\begin{aligned} d &= \frac{2v_i^2\cos^2\theta}{g\cos^2\phi}(\cos\phi\tan\theta - \sin\phi) \\ &= \frac{2v_i^2\cos\theta}{g\cos^2\phi}(\cos\phi\sin\theta - \sin\phi\cos\theta) \\ &= \boxed{\frac{2v_i^2\cos\theta}{g\cos^2\phi}\sin(\theta - \phi)}. \end{aligned}$$

This is exactly the expression required. Note that here we used the trigonometric identity

$$\sin\alpha\cos\beta - \sin\beta\cos\alpha = \sin(\alpha - \beta).$$

(b) The maximum value of d occurs at

$$\frac{\mathrm{d}d}{\mathrm{d}\theta} = 0.$$

One way to solve this derivative is to use the product rule,

$$\frac{\mathrm{d}(f(x)g(x))}{\mathrm{d}x} = \frac{\mathrm{d}f(x)}{\mathrm{d}x}g(x) + \frac{\mathrm{d}g(x)}{\mathrm{d}x}f(x),$$

and apply it to our expression for d in the form

$$d = \frac{2v_i^2\cos\theta}{g\cos^2\phi}(\cos\phi\sin\theta - \sin\phi\cos\theta).$$

We have

$$\begin{aligned}
\frac{\mathrm{d}d}{\mathrm{d}\theta} &= \left[\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{2v_i^2\cos\theta}{g\cos^2\phi}\right](\cos\phi\sin\theta - \sin\phi\cos\theta) + \frac{2v_i^2\cos\theta}{g\cos^2\phi}\left[\frac{\mathrm{d}}{\mathrm{d}\theta}(\cos\phi\sin\theta - \sin\phi\cos\theta)\right] \\
&= \left[\frac{2v_i^2(-\sin\theta)}{g\cos^2\phi}\right](\cos\phi\sin\theta - \sin\phi\cos\theta) + \frac{2v_i^2\cos\theta}{g\cos^2\phi}\left[(\cos\phi\cos\theta - \sin\phi(-\sin\theta))\right] \\
&= -\frac{2v_i^2}{g\cos^2\phi}\sin\theta\,(\cos\phi\sin\theta - \sin\phi\cos\theta) + \frac{2v_i^2}{g\cos^2\phi}\cos\theta\,(\cos\phi\cos\theta + \sin\phi\sin\theta)) \\
&= \frac{2v_i^2}{g\cos^2\phi}\left[-\sin\theta\sin(\theta-\phi) + \cos\theta\cos(\theta-\phi)\right] \\
&= \frac{2v_i^2}{g\cos^2\phi}\cos(2\theta-\phi).
\end{aligned}$$

Note that here we used the trigonometric identity

$$\sin\alpha\sin\beta + \cos\alpha\cos\beta = \cos(\alpha-\beta).$$

We now have to solve this equation

$$\frac{\mathrm{d}d}{\mathrm{d}\theta} = \frac{2v_i^2}{g\cos^2\phi}\cos(2\theta-\phi) = 0,$$

which requires $\cos(2\theta-\phi) = 0$, or $2\theta - \phi = 90^\circ$. Thus the value of θ that maximises d is

$$\boxed{\theta_{\max} = 45^\circ + \frac{\phi}{2}}.$$

Let's plug this into our expression for d, which gives

$$\begin{aligned}
d_{\max} &= \frac{2v_i^2\cos\theta_{\max}}{g\cos^2\phi}\sin(\theta_{\max}-\phi) \\
&= \frac{2v_i^2}{g\cos^2\phi}\cos\left(45^\circ + \frac{\phi}{2}\right)\sin\left(45^\circ + \frac{\phi}{2} - \phi\right) \\
&= \frac{2v_i^2}{\cos^2\phi}\left[\cos 45^\circ\cos\frac{\phi}{2} - \sin 45^\circ\sin\frac{\phi}{2}\right]\left[\sin 45^\circ\cos\frac{\phi}{2} - \cos 45^\circ\sin\frac{\phi}{2}\right] \\
&= \frac{2v_i^2}{g\cos^2\phi}\frac{1}{\sqrt{2}}\left[\cos\frac{\phi}{2} - \sin\frac{\phi}{2}\right]\frac{1}{\sqrt{2}}\left[\cos\frac{\phi}{2} - \sin\frac{\phi}{2}\right] \\
&= \frac{v_i^2}{g\cos^2\phi}\left[\cos^2\frac{\phi}{2} + \sin^2\frac{\phi}{2} - 2\sin\frac{\phi}{2}\cos\frac{\phi}{2}\right] \\
&= \frac{v_i^2}{g\cos^2\phi}\left[1 - \sin\phi\right].
\end{aligned}$$

Note that here we used the trigonometric identity

$$\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = \frac{1}{2}\sin\alpha.$$

Our final result is therefore

$$\boxed{d_{\max} = \frac{v_i^2}{g\cos^2\phi}(1-\sin\phi)}.$$