Physics 101H

General Physics 1 – Honors

Lecture 41 – 11/17/22

Circular Motion

Summary

Topics

Wednesday: Oscillations [chapter 15]

- Mass on a spring
- Simple harmonic motion

Today: Oscillations [chapter 15]

- SHM and uniform circular motion
- Damped oscillations

Announcements

Wednesday November 23: Problem Set 8 due

Next week: Monday's lecture posted to blackboard

Example: Show that the energy of a simple harmonic oscillator is conserved.

Harmonic motion

Uniform circular motion (projected onto a single axis) is secretly simple harmonic motion

In fact, “everything is a harmonic oscillator”

Example: Show that a simple pendulum, in the small angle approximation, undergoes simple harmonic motion.

Problem solving in pairs

Instructions: Solve the following question with a neighbour. Your answers will not be graded; your discussion is for your own learning. It is ok if you do not complete the question, but make sure you identify the key steps and write down the main equations.

Question (a): If you plot the functions $\cos(\omega t)$ and $\cos(\omega t + \pi/4)$, the second curve ($\cos(\omega t + \pi/4)$) will be shifted relative to the first curve ($\cos(\omega t)$). But which way?

(a) left

(b) right

Question (b): If two harmonic oscillators have positions (as a function of time) given by $x_1(t) = \cos(\omega t)$ and $x_2(t) = \cos(\omega t + \pi/4)$, which has the greatest maximum velocity?

(a) $x_1(t)$

(b) $x_2(t)$

(c) neither

Damped oscillations

What happens if we add in friction or other resistive forces? We obtain **damped oscillations**.

Summary

Topics

Today: Oscillations [chapter 15]
- SHM and uniform circular motion
- Damped oscillations

Tomorrow: Waves [chapter 16]
- Forced oscillations
- Types of waves
- Wave equation

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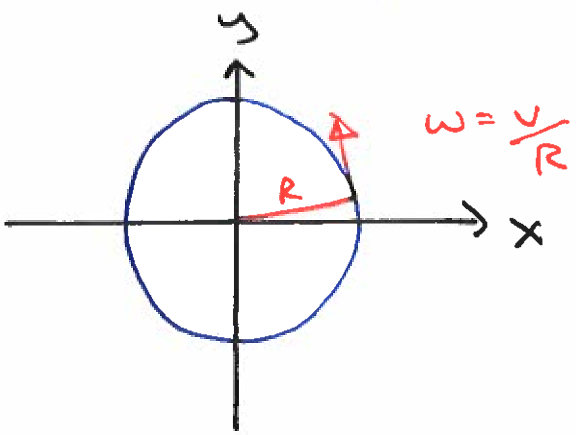
N.B. Energy example at end of notes

PHYSICS 101 - HONORS

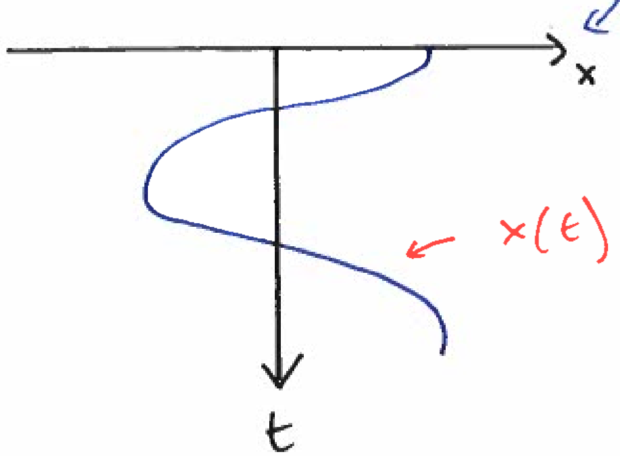
Lecture 41 11/17/22

Uniform circular motion

If we project uniform circular motion onto a single axis, it is equivalent to simple harmonic motion!



"project onto x axis"
=> look at x position as function of time



x position oscillates backwards and forwards in time, exactly like simple harmonic motion

[in fact, so does the y position]

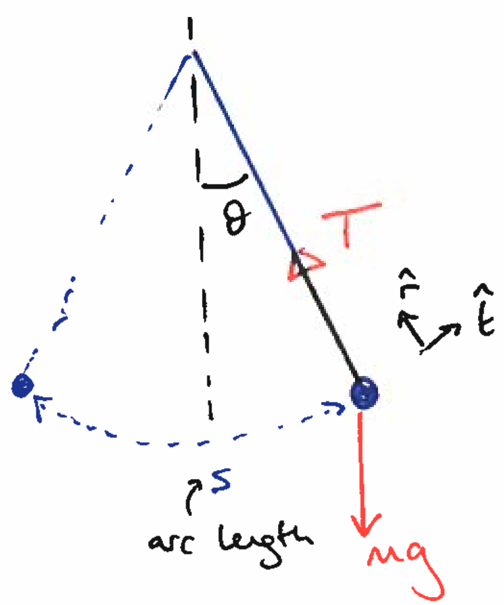
Actually, any "well-behaved" function can be written as an infinite sum of sines and cosines

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

called the <u>Fourier series</u> of a function

The essence of this is that you can break down complicated problems into the sum of a bunch of harmonic oscillators of different frequencies (and amplitudes).

<u>Simple pendulum example</u> (slide 4)



$\bar{T} = T\hat{r}$ and $\bar{F}_g = -mg\cos\theta\,\hat{r} - mg\sin\theta\,\hat{t}$ are the only forces acting on the pendulum

Newton's second law tells us

$$\bar{F}_{net} = m\bar{a}$$

$\Rightarrow \bar{T} + \bar{F}_g = m\bar{a}$ or $T\hat{r} + \left(-mg\cos\theta\,\hat{r} - mg\sin\theta\,\hat{t}\right) = m\bar{a}$

The acceleration in this case is $\bar{a} = a_r\hat{r} + a_t\hat{t}$

⇒ in radial direction:

$$T - mg\cos\theta = m a_r = \frac{mv^2}{r}$$

↑ $a_r = a_c = mv^2/r$

and in the tangential direction

$$-mg\sin\theta = m a_t \quad \Rightarrow \quad a_t = -g\sin\theta$$

But we know that

$$a_t = \frac{d^2 s}{dt^2} \quad \text{and since} \quad s = L\theta \quad \Rightarrow \quad a_t = \frac{d^2}{dt^2}(L\theta) = L\frac{d^2\theta}{dt^2}$$

So we have

$$L\frac{d^2\theta}{dt^2} = -g\sin\theta \quad \Rightarrow \quad \frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

Now introduce $\omega^2 \equiv g/L$ ⇒ $\frac{d^2\theta}{dt^2} = -\omega^2\sin\theta$ ← valid for <u>any</u> angle θ

If we use the Taylor series

$$\sin\theta \simeq \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \ldots$$

then for small angles we can approximate $\sin\theta \approx \theta$

$$\Rightarrow \boxed{\frac{d^2\theta}{dt^2} = -\omega^2\theta}$$

← solution is $\theta(t) = A\cos(\omega t + \phi)$ with $\omega = \sqrt{\frac{g}{L}}$ the angular frequency

↖ But this is equation for simple harmonic motion!

Damped oscillations (slide 6)

So far we have considered a simple oscillator without any friction or resistive force. So let's add in drag (either air resistance, or a spring submerged in fluid)

Now the net force is

$$F_{net} = -kx - bv \quad \Rightarrow \text{Newton's } 2^{nd} \text{ law means}$$

$$-kx - bv = ma$$

$$\Rightarrow -kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

We write this as

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

$$\frac{dx}{dt} = \alpha e^{\alpha t}$$

$$\frac{d^2x}{dt^2} = \alpha^2 e^{\alpha t}$$

Our Ansatz is $x(t) = e^{\alpha t}$

("guess at a solution")

$$\Rightarrow m\alpha^2 e^{\alpha t} + b\alpha e^{\alpha t} + k e^{\alpha t} = 0$$

$$\Rightarrow (m\alpha^2 + b\alpha + k)e^{\alpha t} = 0$$

For this to hold for all times this means

$$m\alpha^2 + b\alpha + k = 0$$

$$\alpha = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} = -\frac{b}{2m} \pm \sqrt{\frac{b^2 - 4mk}{4m^2}}$$

Let's define $\omega^2 \equiv \frac{b^2}{4m^2} - \frac{k}{m}$

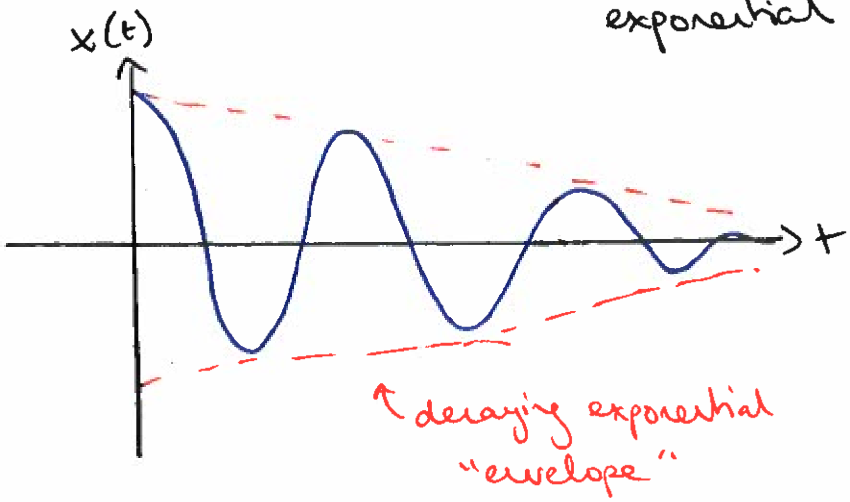
Then $x(t) = e^{-\frac{bt}{2m} \pm \omega^2 t}$

If the damping is relatively small then $k \gg b$ and

$\omega^2 < 0 \Rightarrow \sqrt{\omega^2} = \pm i\omega$

and

$$x(t) = e^{-\frac{b}{2m}t \pm i\omega t} = \underbrace{e^{-\frac{b}{2m}t}}_{\text{decaying exponential}} \underbrace{e^{\pm i\omega t}}_{\text{oscillating term}}$$



$$\omega = \sqrt{\underbrace{\frac{k}{m}}_{\omega_0^2} - \underbrace{\left(\frac{b}{2m}\right)^2}_{\text{damping term}}}$$

ω_0^2 "natural frequency"

$\omega_0 > b/2m$ = "underdamped"

$\omega_0 < b/2m$ = "overdamped"

$\omega_0 = b/2m$ = "critically damped" (no oscillations)

Most general solution is

$$x(t) = \underbrace{Ae^{-bt/2m}}_{\text{damping term}} \underbrace{\cos(\omega t + \phi)}_{\text{oscillating term}}$$

Energy example (slide 6)

Mechanical energy is

$$E_M = E_K + E_p$$

$$= \frac{1}{2} m v^2(t) + \frac{1}{2} k x^2(t)$$

$$= \frac{1}{2} m \left[-A\omega \sin(\omega t + \phi)\right]^2 + \frac{1}{2} k \left[A \cos(\omega t + \phi)\right]^2$$

$$= \frac{1}{2}\left[m A^2 \omega^2 \sin^2(\omega t + \phi) + k A^2 \cos^2(\omega t + \phi)\right]$$

$$= \frac{1}{2}\left[m A^2 \frac{k}{m} \sin^2(\omega t + \phi) + k A^2 \cos^2(\omega t + \phi)\right]$$

$$= \frac{1}{2}\left[k A^2 \sin^2(\omega t + \phi) + k A^2 \cos^2(\omega t + \phi)\right]$$

$$= \frac{1}{2} k A^2 \underbrace{\left[\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)\right]}_{\sin^2(x) + \cos^2(x) = 1}$$

$$= \frac{1}{2} k A^2$$

↖ constant!