Physics 101H

General Physics 1 – Honors

Lecture 26 – 10/21/22

Moment of inertia

Summary

Topics

Yesterday: Angular motion [chapter 10]

- Rigid bodies
- Angular motion
- Moment of inertia
- Torque

Today: Moment of inertia [chapter 10]

- Moment of inertia calculations
- Parallel axis theorem
- Rotational kinetic energy

Announcements

Next week: **No office hours on Monday AFTERNOON**

Two minute essay

Instructions: Draw a diagram for the following topic. You have two minutes. You may not use your notes and you should not consult with others around you. Your answer will not be graded; your answer is for your own learning and you don't need to share your answer.

Question: Construct a table with two columns, one for linear/translational motion and one for rotational motion, that summarises all the relevant quantities, kinematic equations and dynamics equations for each case. Each row of your table should be one pair of quantities or equations (e.g. displacement and angular displacement). Once you have finished, compare your table with your neighbours.

Example: Find the moment of inertia for a rectangular plate that rotates about an axis through its center of mass.

Parallel axis theorem

So far we have only considered the moment of inertia around an axis that passes through the center of mass – but what about other axes?

The **parallel axis theorem** provides one way to determine this (relatively) easily

Rotational kinetic energy

Why all the fuss about the moment of inertia?

We have seen that it is the rotational analog of mass – and this analogy goes further!

Moment of inertia is central to rotational kinetic energy

Summary

Topics

Today: Moment of inertia [chapter 10]

- Moment of inertia calculations
- Parallel axis theorem
- Rotational kinetic energy

Next week: Angular motion [chapters 10/11]

- Rolling motion
- Angular momentum
- Rotational dynamics

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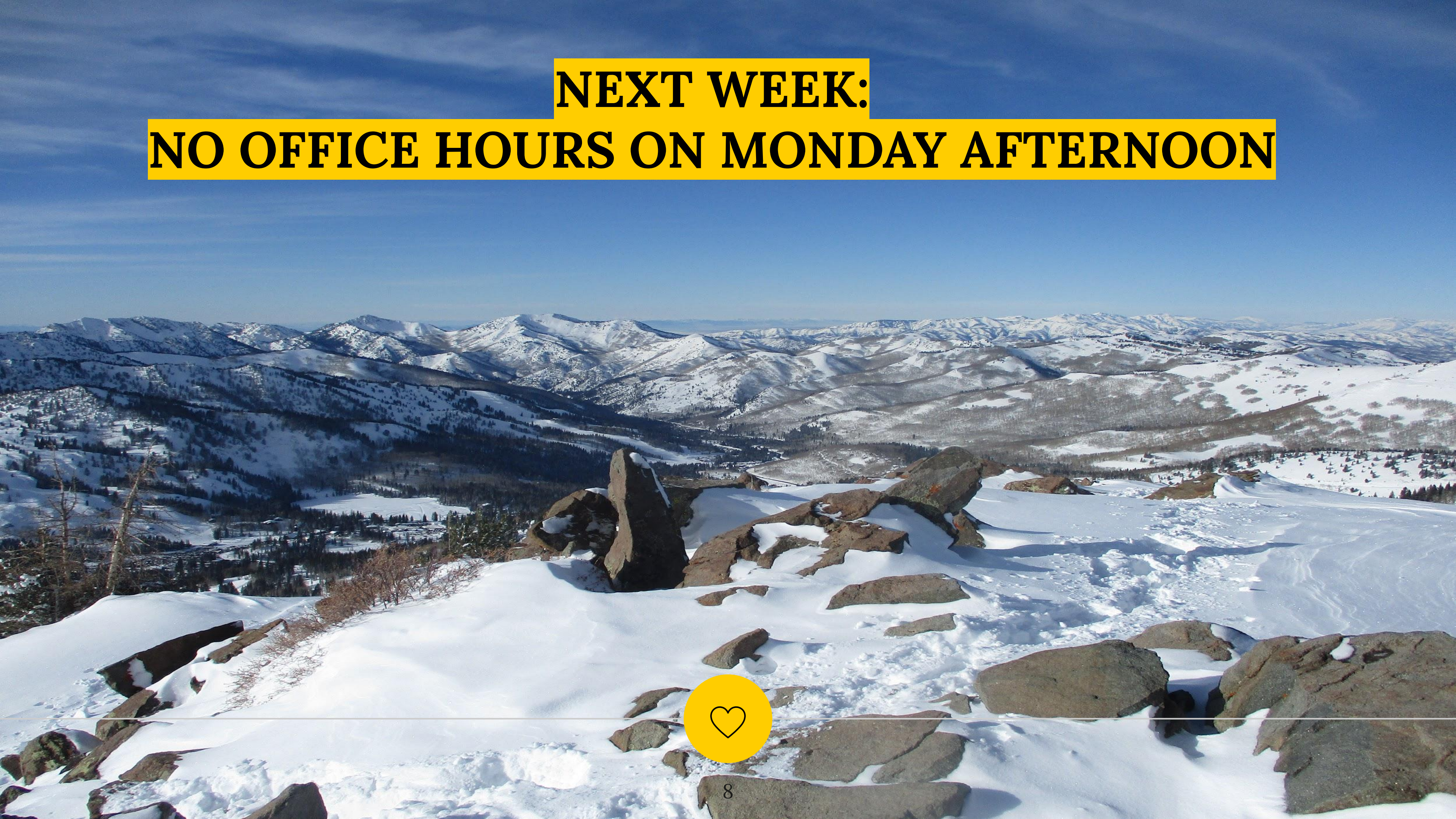
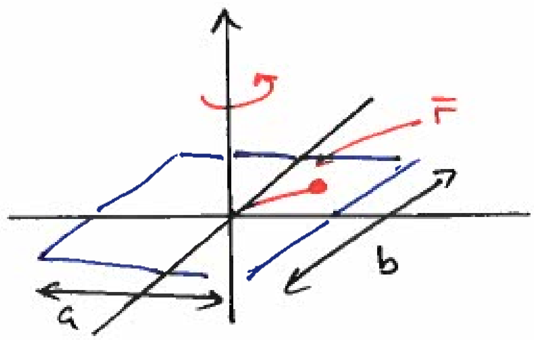


PHYSICS 101 - HONORS

Lecture 26 10/21/22

N.B. sphere example on last page of notes

Plate example (slide 4)



$$I = \int r^2 \, dm$$

Density is $\rho = \frac{M}{V} = \frac{dm}{dV}$

$$\Rightarrow \quad dm = \rho \, dV = \rho \, dx \, dy \, dz$$

assume we can ignore for a thin sheet

$$I = \int_{-\frac{b}{2}}^{b/2} \int_{-a/2}^{a/2} r^2 \rho \, dx \, dy$$

$$= \rho \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} (x^2 + y^2) \, dx \, dy$$

$$= \rho \left[\int_{-b/2}^{b/2} dy \int_{-a/2}^{a/2} x^2 \, dx + \int_{-b/2}^{b/2} y^2 \, dy \int_{-a/2}^{a/2} dx \right]$$

$$= \rho \left[y \Big|_{-b/2}^{b/2} \frac{x^3}{3} \Big|_{-a/2}^{a/2} + \frac{y^3}{3} \Big|_{-b/2}^{b/2} x \Big|_{-a/2}^{a/2} \right]$$

$2^3 = 8$

$$= \rho \left[\underbrace{\left(\tfrac{b}{2} - \left(-\tfrac{b}{2}\right)\right)}_{b} \underbrace{\left(\frac{a^3}{3 \cdot 8} - \frac{(-a)^3}{3 \cdot 8}\right)}_{= \frac{a^3}{12}} + \underbrace{\left(\frac{b^3}{3 \cdot 8} - \frac{(-b)^3}{3 \cdot 8}\right)}_{\frac{b^3}{12}} \underbrace{\left(\tfrac{a}{2} - \left(-\tfrac{a}{2}\right)\right)}_{a} \right]$$

$$= \rho \left(\frac{b a^3}{12} + \frac{a b^3}{12} \right)$$

$$= \frac{\rho}{12} a b (a^2 + b^2)$$

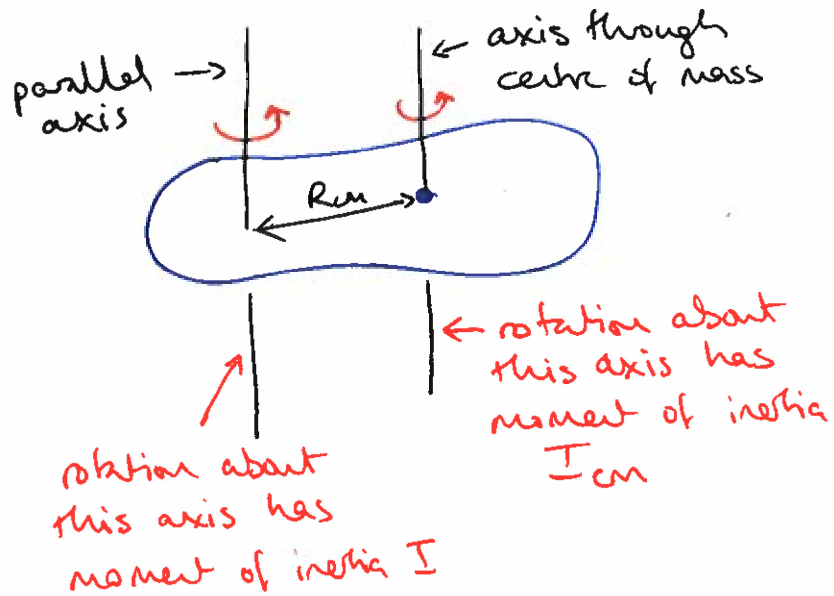
But $\rho = \frac{M}{V} = \frac{M}{ab}$

$$\Rightarrow I = \frac{M}{ab} \cdot \frac{1}{12} \cdot ab(a^2 + b^2)$$

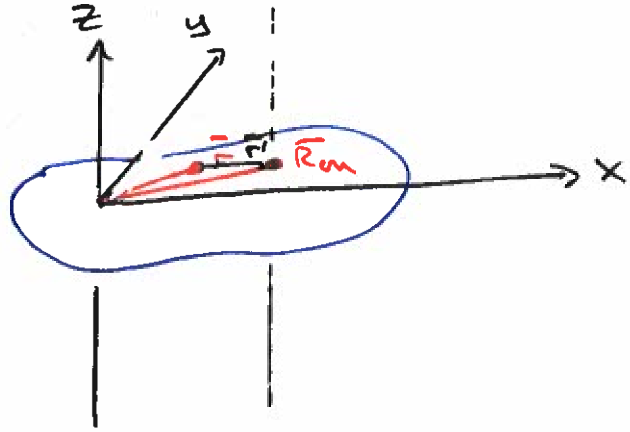
$$\boxed{= \frac{M}{12}(a^2 + b^2)}$$

Parallel axis theorem (slide 5)

Theorem: $I = MR_{cm}^2 + I_{cm}$ for any axis parallel to an axis through the centre of mass and at a distance R



To prove this, note that I is independent of the distribution of mass in the direction of the axis, which we call the z-axis -> look at volume in (xy) plane



$$I = \int r^2 dm = \int (x^2 + y^2) dm$$

$$\bar{r} = \bar{R}_{cm} + \bar{r}'$$

$$\Rightarrow x = x_{cm} + x'$$
$$y = y_{cm} + y'$$

$$\Rightarrow I = \int (x_{cm}^2 + 2x'x_{cm} + x'^2 + y_{cm}^2 + 2y'y_{cm} + y'^2) dm$$

$$= \int (x_{cm}^2 + y_{cm}^2) dm + \underbrace{2\left(\int x_{cm} x' dm + \int y_{cm} y' dm\right)}_{=0}$$

$= 0$ because mass is evenly distributed around centre of mass

$$+ \int (x'^2 + y'^2) dm$$

$$= R_{cm}^2 \int dm + \int r'^2 dm$$

$$= M R_{cm}^2 + I_{cm} \qquad \#$$

Rotational kinetic energy (slide 6)

Translational

$$E_K = \frac{1}{2} M v^2$$

$$W = \Delta E_K$$

Rotational

$$E_K^{rot} = \frac{1}{2} I \omega^2$$

$$W^{rot} = \Delta E_K^{rot}$$

Comparison table (slide 3)

Translational

Position $\bar{r}$

Velocity $\bar{v} = \frac{d\bar{r}}{dt}$

Acceleration $\bar{a} = \frac{d\bar{v}}{dt} = \frac{d^2\bar{r}}{dt^2}$

$\bar{v} = \bar{a}t + \bar{v}_0$

$\bar{x} = \frac{\bar{a}t^2}{2} + \bar{v}_0 t + \bar{x}_0$

Mass m

Rotational

Angular displacement $\bar{\vartheta}$

Angular velocity $\bar{\omega} = \frac{d\bar{\vartheta}}{dt}$

Angular acceleration

$\bar{\alpha} = \frac{d\bar{\omega}}{dt} = \frac{d^2\bar{\vartheta}}{dt^2}$

$\bar{\omega} = \bar{\alpha}t + \bar{\omega}_0$

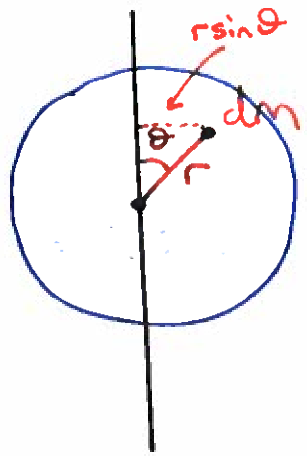
$\bar{\vartheta} = \frac{\bar{\alpha}t^2}{2} + \bar{\omega}_0 t + \bar{\vartheta}_0$

Moment of inertia I

$I = \int r^2 dm$

Sphere example (AKA why I was wrong yesterday)

Yesterday I tried to calculate I_{sphere}. I was almost correct, but I forgot an important point - I got the "r^2" wrong! I forgot that we need the distance of the infinitesimal mass dm from the z axis not the origin! Don't I is defined with respect to an axis - in this case we define it to be the z axis.



The "r^2" in $I = \int r^2 dm$ is the distance to the axis, defined in a way to be perpedicular to the axis.

Thus we need

$$I = \int (r \sin\theta)^2 dm$$

$$dm = \rho\, dV = \rho\, r^2 dr \sin\theta d\theta d\phi$$

$$\Rightarrow I = \int (r\sin\theta)^2 \cdot \rho r^2 dr \sin\theta\, d\theta\, d\phi$$

$$= \rho \int_0^R r^4 dr \int_0^\pi \sin^3\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= \rho \cdot \frac{R^5}{5} \cdot \left(-\frac{3}{4}\cos\theta + \frac{1}{12}\cos(3\theta)\right)\Big|_0^\pi \cdot 2\pi$$

$$= \rho \cdot \frac{2\pi R^5}{5} \cdot \left(\frac{2}{3} - \left(-\frac{2}{3}\right)\right) = \rho \cdot \frac{8\pi R^5}{15}$$

Now we use

$$\rho = \frac{M}{V} = \frac{M}{\left(\frac{4\pi R^3}{3}\right)} = \frac{3M}{4\pi R^3}$$

So

$$I = \frac{3M}{4\pi R^3} \frac{8\pi R^5}{15}$$

$$= \boxed{\frac{2MR^2}{5}} \qquad \#$$

Some comments:

- Different ways to do this include integrating over a disc and then integrating that vertically (with decreasing radius) or integrating over a thin shell then integrating over shells of different radius

- Spherical coordinates are (r, ϑ, ϕ), one angle runs from 0 to π and the other runs from 0 to 2π.

mathematicians use a convention where ϑ and ϕ are switched!

- In Cartesian (rectilinear) coordinates, the "volume element" dV is $dV = dx\,dy\,dz$.

In spherical coordinates $dV = r^2 dr \sin\vartheta \, d\vartheta \, d\phi$

This can be obtained from

$$x = r \sin\vartheta \cos\phi$$
$$y = r \sin\vartheta \sin\phi$$
$$z = r \cos\vartheta$$

see the wikipedia article "Spherical coordinate system"

The volume elements are related by the <u>Jacobian</u> of the coordinate transformation $(x, y, z) \to (r, \vartheta, \phi)$

$$dV\big|_{\text{spherical}} = |J| \, dV\big|_{\text{cartesian}}$$

where $J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \vartheta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix}$ is the <u>Jacobian matrix</u>

and $|J|$ is the <u>determinant</u> of J. Recall the determinant of a 3×3 matrix is

$$\begin{aligned} |M| = \; & M_{11}\left(M_{22}M_{33} - M_{23}M_{32}\right) \\ & - M_{2}\left(M_{21}M_{33} - M_{23}M_{31}\right) \\ & + M_{3}\left(M_{21}M_{32} - M_{22}M_{31}\right) \end{aligned}$$

where $M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}$